Government

SAHAR

For

in the

Adjudged to

T. C. SANYAL, B.A.,
Head Master.

Annual Examination 189

SHUBHANKARI or INDIGENOUS BENGALI ARITHMETIC
14th Edition. 8 as
KEY TO THE ABOVE. Re. 1

ELEMENTS OF EUCLID.

Our knowledge concerning external objects is grounded entirely on the information received through the medium of the senses. The science of physics considers bodies as they exist, invested at once with all their various qualities, and endowed with their peculiar affections. Hence its researches are directed by that refined species of observation which is termed *experiment*. Geometry takes a more limited view, and contemplates merely the forms which bodies present, and the spaces which they occupy. In considering an external object, we can by successive acts of abstraction reduce the complex idea which arises in the mind into others which are progressively simpler. *Body*, divested of its essential characteristics, presents the mere idea of *surface*; a surface considered apart from its peculiar qualities, exhibits only *linear boundaries*; and a line, abstracting its continuity, leaves nothing in the imagination but the *points* which form its boundaries.

The *Science of Geometry* treats of figured space or extension.

Figured space or *extension* is of three kinds, *lines*, *surfaces* and *solids*, and these are of one, two or three dimensions respectively.

Thus, a Solid has *length*, *breadth* (or width), and *thickness* (or depth or height); a Surface has *length* and *breadth*; a *Line* has only *length*.

OPINIONS.

I made up my mind some time ago in view of probable retirement from the Education service to discontinue giving opinions on educational works. I can only say the limits of subject for Entrance Examination have not been changed during past four years. I see no reason why your Euclid should not be adopted by board of studies as an alternative text-book in subject.

2nd September 1888. JOHN ELIOT, M A

I have looked through Mr P. Ghosh's new edition of the Elements of Euclid and I am glad to be able to say that the large amount of new matter judiciously introduced in this edition really makes it an admirable text book. If the plan of appointing text books for the Entrance Examination be finally settled upon, it will be but bare justice to Mr Ghosh's edition to mention it as an alternative text book along with the two editions that have been already named.

2nd September 1888. ASUTOSH MUKHOPADHYAY,
M A, F R A S, F R S (Edin)

I have looked through your 'Elements of Algebra' and think it useful and well adapted to the purpose for which it is intended. The examples are numerous and good.
A W CROFT, M A
Director of Public Instruction, Bengal

"I have looked through your edition of Wood's Elements of Algebra remodelled so as to adapt it for the use of native students. I have much pleasure in expressing my favourable opinion of its merits as a text book. It gives fully and clearly all that is required to assist native students to pass the ordinary University Examinations in Algebra. The examples are numerous and well selected. Solutions are also given freely to aid the student. These, I may add, are neatly and methodically worked out. A complete collection of the Calcutta University Examination Papers in Algebra makes this work doubly valuable to an Indian Student."

It therefore seems to me to give all that either Entrance or First Arts candidates can possibly require and as it has the great additional merit of cheapness, it will, I have no doubt, supersede Todhunter's Algebra very largely in Indian Schools and junior College classes."

"So far as I can judge, it (Euclid IV Books) appears to me to be very well-adapted for the purposes you have in view. It is moderately priced. It gives all that an *Indian student can require for the Entrance Examination.*"

JOHN ELIOT, M. A.
Senior Professor of Mathematics, Presidency College, Calcutta

"I have looked through your edition of Wood's Algebra, and am of opinion that it is a really useful work, very well adapted to the needs of Indian students. Its special value lies in the unusually copious sets of examples, which are, as far as I can see, well selected and carefully arranged. It will also greatly assist the student to find such a large number of examples carefully worked out in the text by methods that seem to be generally clear and instructive."

I am glad also to be able to speak in very high terms of your Elements of Euclid. The Notes which you add to each book convey a great amount of very interesting information on geometrical points, and I do not know of any other edition of Euclid in which the student will find so large a collection of useful deductions. Your hints for solution of these seem also very judicious.

DR. HUGH W. M'CANN, M. A.
Professor of Mathematics Presidency College, Calcutta.

SMALLER SHUBHANKARI. 11th. Edition. 4 as.
SARALA SHUBHANKARI 6th. Edition. 2 as.
SARALA PARIMITI. 5th Edition. 2 as. 6 p.
KEY TO THE ABOVE. 4 as.
DHARAPATA or SHUBHANKARI PROBESHIKA. 1 anna
NAVA DHARAPATA (paper) 1 anna, (board) 1 anna 6 p.

पाटीगणित (हिन्दी)। कपड़ेकी जिल्द १।≠) कागजकी १)
सहज पाटीगणित (हिन्दी)। कपड़ेकी जिल्द ॥) कागजकी ।≠)
पाटीगणित समाधान (हिन्दी)। अभी छप रहा है।

علم حساب (اردو) کپڑے کی جلد ۱ روپیہ ۴ آنہ کاغذ کی جلد ۱ روپیہ ۳ آنہ

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DR. HUGH W. M'CANN, M. A.
Professor of Mathematics Presidency College, Calcutta.

"I have looked through the second part of your Algebra, and think it is quite up to the standard of the First Art. It ought to be very useful to candidates for the F A Examination"

ALFRED M. NASH, M A.

Professor of Natural Philosophy and Astronomy, Presidency College, Calcutta

"I have examined the editions of Euclid's Elements and of Wood's Algebra which Mr P Ghosh has specially prepared to meet the requirements of Indian students"

"The large number of useful and judiciously selected problems and exercises which are worked out to assist the students in understanding the subject and the hints to the solution of exercises and especially of the Calcutta University Examination Papers which are added, materially enhance the value of these publications as text books for those who are preparing for their degree examinations

JOHN HARDIE, M A.,

Professor of Mathematics, Doveton College, Calcutta.

"I have carefully examined the second part of your Elements of Algebra, and I think it likely to prove useful for students to work out with ease and neatness all problems that may be given them within the scope of the First Examination in Arts"

D VAN IMPE S J,

Rector and Professor of Mathematics, St Xavier's College, Calcutta.

"I have read your Algebra, part II, and I have much pleasure to say it will form an excellent text book for the F A students The explanations of the articles are lucid, and the examples are judiciously selected, and are more numerous and varied than can be found in any other book The examples worked out in the book and the Calcutta University Examination Papers given at the end have doubly added to its utility On the whole I think your book will prove a better text for the F A students than even Todhunter's Algebra

GOURY SUNKER DEY, M A

Professor of Mathematics, General Assembly's Institution, Calcutta

"I have looked through your Elements of Algebra," and have to say with great pleasure that it is a nicely got up work which does you great credit The principles have been clearly explained and very well illustrated by the examples worked out at the end of each chapter The exercises have been very judiciously and copiously selected I think the book may be very profitably adapted as a text book in our higher class schools and colleges"

BAIDY NATH BASU M A

Professor of Mathematics, Metropolitan Institution, Calcutta

"I am very glad that you have completed your Elements of Algebra by bringing out Part II early this session I find that Part II is like Part I admirably suited to the requirements of those for whom it is intended The exposition of principles is more lucid and the examples and solutions more numerous than are found in any of the treatises on Algebra used as text books in our schools and colleges I believe no Mathematical teacher that should carefully examine your Elements would ever hesitate to adopt it as a text book

GANGADHAR BANERJEE, M A.,

Professor L M S College, Bhowanipore

"I have looked through the second part of Mr Ghosh's *Elements of Algebra* and have much pleasure to state that the work is well adapted to the requirements of students preparing for the F A Examination of the Calcutta University The principles are clearly explained and numerous examples have been worked out in illustration of the principles The book contains a large collection of well selected examples for exercise, which will be of great use to students."

S C GUI, M A,

Lecturer, Sanskrit College.

"I have looked through the book [Algebra Part II] and consider it useful to students preparing for the Examination in First Arts, especially as it contains a variety of examples judiciously collected. I have recommended it to the students of the Berhampore College for using it as a text-book.'

HARIDAS GHOSH,

Professor of Mathematics, Govt College, Berhampore.

EUCLID'S
ELEMENTS OF GEOMETRY

CONTAINING

PROBLEMS AND THEOREMS ON MODERN GEOMETRY

WITH

HINTS FOR THE SOLUTION OF EXERCISES,

BY

P. GHOSH,

AUTHOR OF "ELEMENTS OF ALGEBRA," "ELEMENTS OF ARITHMETIC"
"ELEMENTS OF MENSURATION," "ELEMENTS OF
TRIGONOMETRY," &c &c

PART I

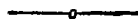
THE FIRST FOUR BOOKS OF THE ELEMENTS

SIXTEENTH EDITION

REVISED AND ENLARGED

BY HIS SON

A S GHOSH, F R. A S (LONDON),

Professor of Mathematics and Economics,
City College, Calcutta

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1895.

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28 CONVENT ROAD—CALCUTTA

PREFACE TO THE SIXTEENTH EDITION

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It was with some reluctance that I have been persuaded to make any changes in this Edition of my father's *Elements of Euclid*, for the work was already so admirably complete that it seemed to me like "carrying coals to Newcastle" Considering however the present tendency to modernize Euclid more and more, I have been carried irresistibly by the tide of mathematical opinion to make such alterations and additions as seemed needed As an illustration of the former it many be mentioned that the cumbrous and redundant language of Euclid has been to some extent abandoned in favour of the modern abbreviated method so popular with students for facilitating their grasp of the subject by lessening the work of the memory But it was not considered desirable to do so too early for it would only result in puzzling the very young student, thereby destroying its *raison d'être*, it has been done gradually, at first tentatively and then only more rigorously Hence the First Book has been left intact, that the very language of the great master of antiquity might form the ground-work of reasoning in the beginner's mind In the Second Book this abbreviated method has been introduced, but very cautiously and with much admonitions For the unwary author, in his anxiety to avoid a Charybdis of superfluous or defective language, is but too often shattered against a Scylla by not warning the student against proving the Propositions of Book II by purely algebraical formulas From the Third Book however the process of abbreviation has been worked out more completely as by that time the student is probably able to differentiate between the *essence* of a geometrical proof and the mere language in which it might be clothed

As regards the additions made in this issue, they are so numerous that it would only weary the reader to go into details here, suffice it then to give a general idea of them in the briefest manner possible They are sometimes by way of extensions and generalizations, sometimes in the nature of limitations By this method I thought to awaken in the student the power of drawing

inferences, and to enable him to acquire a keen perception of analogies—the two essentials of an enquiring and inventive mind. Just to give a few instances of such extensions and limitations, the new matter added in the Notes at the end of each Book may be mentioned. Of course as might be expected these are less striking in the First Book than in the later ones. The most characteristic instances however would perhaps be the NOTES and OBS. at the end of many Propositions. The following are a few taken at random :—those in p. 267, 268, 280, 284—285, 290, 318, 376, 377, (COR. p. 379), 391, 392, etc.; perhaps the most interesting will be found in p. 330 as it is, I believe, not published elsewhere. As regards the many new Propositions, may be mentioned briefly the one in pp. 191–192, most of those in pp. 288—290, those in pp. 378, 379, the new proof of the Nine Point Circle (perhaps the shortest) in pp. 380–381, and the very important Prop. in pp. 384–387.

Some apology may be needed for Prop. X. p. 354 *(To find the length of the circumference of a circle in terms of the diameter)*, and Prop. XI. p. 357 *(To find the area of a circle in terms of the radius)*. Although both are of the highest importance, they are never given in works on Elementary Geometry. I confess I do not know the reason why; perhaps because it was deemed impossible to give solutions at once elementary and Geometrical. As to whether the solutions given in this treatise are elementary or not, I leave it to the student to decide; for myself I am inclined to believe they are. But that they are rigorously Geometrical is, I trust, unquestionable. Of course objections might be raised against the abbreviations employed, *i.e.*, the supposition 'Let the radius $= r$'. But it is easier to write 'r' than "the radius", "r^2" than 'the sq. on the radius', '$r \cdot AB$' than 'the rectangle contained by the radius and the chord AB", especially as the student has been warned repeatedly in the Introduction to Book II and in the Notes at the end of Book II that these are *verbal abbreviations* only, and do not imply Arithmetical or Algebraical formulas. Again, it might be objected that "$\because AB^2 = 3r^2$, it does not follow Geometrically that "$\therefore AB = r\sqrt{3}$". But surely this is no more than the purely Geometrical inference "$\because$ the sq. on PQ = the sq. on RS, $\therefore$ PQ = RS". As to the numericals involved ($\sqrt{3}$, etc.), they occur purely incidentally, and in no way detract from the rigorously orthodox nature of the solution. Nay, the very fact that they are incommensurable is in itself a *prima fácie* evidence of their geometrical character. If further corroborative evidence were needed, I may mention that it is not unknown in Euclid to have "a line the square on which is three times the square on another line" (*i.e.* AB $= r\sqrt{3}$). But enough. I trust to obtain a verdict, as probably the case has been already proved to the hilt. I may mention however in passing that the NOTE in p. 356 is merely an arithmetical illustration, and therefore "like the flowers that bloom in the spring, has nothing to do with the case"—I mean it is only an interesting application, but not an integral part, of the Proposition itself.

By far the most important, and perhaps the most unique portion of this issue is the "Theory of Maxima and Minima" (p 405) In writing this section I have been bold, I trust not too bold, for I have attempted to bring down one of the highest conceptions of the Differential Calculus to the level of the mind of the beginner by means of the simplest analogies How far I have succeeded in this endeavour, I leave it to the lenient judgment of my critics, to decide Suffice it to say that I lay no claim to originality in the methods of Class B and C, pp 418, 426—for I have only done the work of an expositor I venture to submit however that the whole theory as laid down and exemplified from p 405 to p 417 is not to be found elsewhere To illustrate the theory more effectually I shall be thankful for any suggestions or contributions of problems from my readers

A practical improvement has been made in this issue an Index of Definitions (other than those at the beginning of each Book) From this it will be seen that I have taken the liberty of coining or adapting three new definitions ("Antipedal Triangle" 'Images' 'Point of Reflexion') for obvious advantages It may also be remarked that most of the older diagrams have been replaced, and the general appearance of the book made more attractive

How far I have been indebted to others for the changes made in this Edition, I cannot tell But one thing I do know, that were it not for the fostering care of my father my early training in Mathematics would indeed have been void He granted me the privilege of helping him to issue his improved Edition of this treatise in 1888, and now it is to me a melancholy pleasure to edit his works - but for the consciousness that his mantle has fallen on unworthy shoulders I must also acknowledge my obligation to my late Professor, MR J EDWARDS, M A, formerly Fellow of Sidney Sussex College, Cambridge, whose well known work on the Differential Calculus is such a favourite both in England and in India It was in the course of his lectures on this subject that a few words dropped from his lips from which I conceived the plan of the "Theory of Maxima and Minima" mentioned above Lastly, though by no means the least, my warmest thanks and sincerest gratitude is due to my friend MR ANDREW CLAUDE DE LA CHEROIS CROMMELIN, F R A S, of the Royal Observatory, Greenwich, whose kindly encouragement and help has enabled me to undertake a task by no means light Let this suffice here, I shall reserve a fuller acknowledgment of his assistance for another treatise

CALCUTTA,
11th March, 1895.

A. S G.

PREFACE TO THE ELEVENTH EDITION.

In this edition I have added more Questions and Riders and some Additional Propositions which serve as an introduction to Modern Geometry. Here I acknowledge with thanks that, in preparing this new edition, I have received considerable assistance from A. M. Nash Esq., M.A., and many important suggestions from the Hon'ble Dr. Guru Das Banerjee M.A. and A. M. Bose Esq., M.A., of Christ's College, Cambridge.

Calcutta,
18th August 1888.

P. G.

PREFACE TO THE SECOND EDITION

In this edition, at the request of several educational authorities, numerous easy deductions have been added as riders to almost every Proposition of the Elements.

Calcutta,
18th November, 1880.

P. G.

ORIGINAL PREFACE

This edition of Euclid has been prepared with the special object of meeting the requirements and difficulties of students preparing for the Entrance Examination of the Indian Universities. The text of Euclid as given in Dr. Simson's Edition has been generally followed, with copious explanatory Notes and Questions. To these has been added a collection of many problems and theorems, with analytical and synthetical solutions. To make the work of special interest to Entrance Candidates a large collection of exercises taken chiefly from the Calcutta, Madras, Bombay, Cambridge and other University Papers, has been given, with Hints for Solution. As the solution of a problem is more interesting than that of a theorem, of the [illegible] exercises added more than 250 are problems. It will be seen therefore, that in this work the problems are more numerous than in any other work of the same kind. The "Hints for Solution" will serve as a key to the exercises—with directions for the construction of diagrams of all the problems and theorems, which may require them. The utmost attempt has been made to render the book both useful and comprehensive.

The author will be most happy to receive from Mathematical Teachers any suggestions conducive to make the work as useful and complete as possible.

P. G.

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BOOK I.

BOOK II.

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Appendix

N B —The *Notes* at the end of each Book might be read simultaneously with the text

INDEX OF DEFINITIONS

[*Other than those at the beginning of each Book.*]

BOOK I.

DEFINITIONS.

1. A **point** has position, but it has no magnitude

2. A **line** has position and it has length, but has neither breadth nor thickness

3. The extremities of a line are *points*, and the intersection of two lines is a *point*

4. A **straight line** is that which lies evenly between its extreme points

5. A **surface** has position, and it has length and breadth, but no thickness The extremities of a surface are *lines*

6. A **plane surface** is that in which any two points being taken, the straight line between them lies wholly in that surface

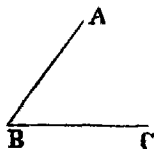
7. A **plane figure** may be composed of points or lines or of points and lines in a plane

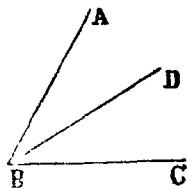
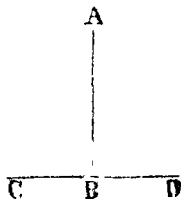
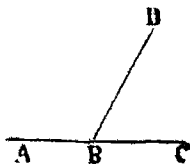
8. If three or more points lie in a straight line, these points are said to be **collinear.**

9. When two straight lines meet each other, their inclination or opening is called a **rectilineal angle.**

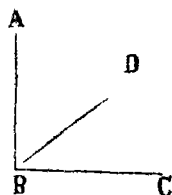
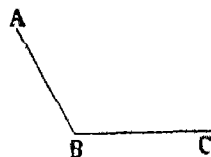
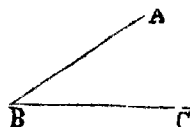
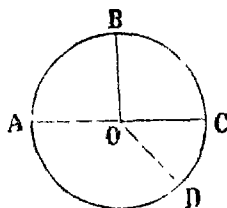
For shortness the term *angle* may be used for the phrase *rectilineal angle*.

10 The two straight lines which make, form or contain an angle, are called the **sides** or **arms** of the angle, and the point at which the arms meet is called the **vertex** of the angle

Let *AB*, *CB* meet each other at the point B, the angle is sometimes designated simply by the letter at the vertex, as the angle B, sometimes by three letters ABC, or CBA, the letter at the vertex being placed always in the middle

Angles, like all other quantities are susceptible of addition subtraction, multiplication and division

Thus the angle ABC is the sum of the angles ABD and DBC, and the angle ABD is equal to the difference of the angles ABC and DBC

11 When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a **right angle**, and the straight line which stands on the other is called a **perpendicular** to it

12 When the sum of two angles is such that the two arms which make the compound angle are in a straight line each of the angles is called the **supplement** of the other, and the angles are said to be **supplementary angles.**

If AB and BC be in a straight line, ABD and DBC are supplementary angles

13 When the sum of two angles is a right angle, each is called the **complement** of the other; and the two angles are said to be **complementary angles.**

If ABC is a right angle, ABD and DBC are complementary angles

14 An **obtuse angle** is that which is greater than a right angle

An **acute angle** is that which is less than a right angle.

15 A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another

16 That point in a circle, from which straight lines drawn to the circumference are all equal, is called the **centre** of the circle

17 A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference

18 A **radius** of a circle is a straight line drawn from the centre to the circumference.

19 A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

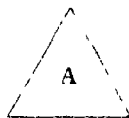
20 **Trilateral** figures, or **triangles** are those which are contained by three straight lines.

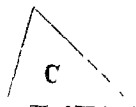
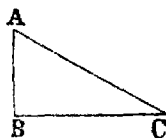
21 **Quadrilateral,** or **four-sided** figures are those which are contained by four straight lines.

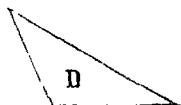
22 **Multilateral** figures, or **polygons** are those which are contained by more than four straight lines.

23 A polygon which has five sides is called a **pentagon,** a polygon which has six sides is called a **hexagon,** and so on.

Of three-sided figures.

24 An **equilateral** triangle is that which has three equal sides.

25 An **isosceles** triangle is that which has two sides equal.

26 **A scalene** triangle is that which has three unequal sides.

27 A **right-angled** triangle is that which has a right angle.

28 The side opposite to the right-angle in a right-angled triangle is called the **hypotenuse.**

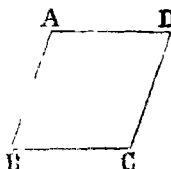
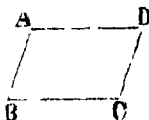
29 An **obtuse-angled** triangle is that which has an obtuse angle.

30. An **acute-angled** triangle is that which has three acute angles

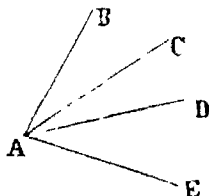
E

Of four-sided figures.

A D
B C

31. A **square** is that which has all its sides equal, and all its angles right-angles

32. A **rhombus** or **lozenge** is that which has all its sides equal but its angles are not right angles

33. A **rhomboid** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles

34. The line joining any two opposite angles of a quadrilateral is called a **diagonal.**

35. If three or more straight lines pass through a point, these lines are said to be **concurrent.**

36. A system of four or more concurrent straight lines is called a **pencil** of lines, each line is called a **ray.**

EXPLANATION OF TERMS EMPLOYED IN GEOMETRY

An **axiom** is a self-evident truth, which admits of no demonstration

The questions and truths of geometry stated and considered in small separate discourses are called **propositions.**

A proposition is a **problem** when some geometrical construction is required to be effected The letters **Q. E. F.** placed at the end of problems stand for **quod erat faciendum,** that is, *which was to be done*

A proposition is a **theorem** when some geometrical truth is rendered evident by means of a train of reasoning called a *demonstration* The letters **Q. E. D.** placed at the end of theorems stand for **quod erat demonstrandum,** that is, *which was to be demonstrated*

Every proposition consists of two parts —

1 When the proposition is a problem, the two parts are the **data** (or things given) and **quæsita** (or things required)

2 When the proposition is a theorem, the two parts are the **hypothesis,** or that which is assumed and the **conclusion,** or that which is asserted to follow therefrom

One proposition is the **converse** of another when the *hypothesis* of the latter becomes the *conclusion* of the former and *vice versa.*

If A is B, then C is D (i)

The following is the converse of the above —

If C is D then A is B (ii)

From the above two, we may infer two others called their *contrapositives*

Of (i), if C is not D then A is not B (iii)

Of (ii), if A is not B, then C is not D (iv)

(iv) is the *obverse* of (i) and (iii) is the *obverse* of (ii)

A **postulate** is a problem, the possibility of which is admitted to be self-evident

A **lemma** is a subsidiary truth, employed for the demonstration of a theorem or the solution of a problem.

A **corollary** is a theorem which is deduced from the demonstration of a proposition.

A **scholium** is a remark on one or several preceding propositions which tends to point out their connexion, their use, their restriction or their extension.

Figures which are equal **in every respect,** that is, which can be made to coincide, are said to be **identically equal.** Such figures are also called **congruent figures.**

The straight line which cuts a system of lines, a circle or any other geometrical figure is called a **secant** or **transversal.**

Superposition is the process by which one magnitude may be conceived to be placed upon another, so as exactly to cover it, or so that every part of the one shall exactly coincide with the corresponding part of the other. When a body is superposed upon another, it is said to be **applied** to it.

The method of superposition assumes that a figure may be moved in space, turned over, and put down in another place, without change of shape or size, that is without changing the relative positions of its boundaries. Thus

1. Two straight lines are equal, if one straight line be placed on the other so that one end of the one falls on one end of the other, then the other ends will coincide.

2. Two angles are equal, if one angle be placed on another so that the vertex of the first falls on the vertex of the other and one arm of the first falls on one arm of the other, then the remaining arm of the first will coincide with the remaining arm of the other.

3. Two circles have equal radii, if the centre of one circle be placed on the centre of the other, the circumference of the first will fall on the circumference of the other. For every point on the circumference of each circle is equally distant from the common centre.

OBS. Hence circles which have equal radii are identically equal.

RULE OF IDENTITY.—If there is but one A and one B, then, from the fact that A is B, it necessarily follows that B is A.

POSTULATES

Let it be granted that

1 A straight line may be drawn from any one point to any other point

Let A and B be any two points, the straight line joining AB is called the *join* of A, B AB is a finite straight line

2. A terminated straight line may be produced to any length in a straight line —

To produce AB, means that the straight line is to be produced from the extremity B, to produce BA, means that the line is to be produced from the extremity A

To use the straight edge of a ruler is admitted in the first postulate, as well as in the second

3 A circle may be described from any centre, at any distance from that centre

Euclid admits in this postulate the use of a pair of compasses to describe a circle But it is the *circumference* of the circle that is really drawn by the compasses, though the enclosed figure is the circle required Hence it is convenient to make the following convention the word 'circle' will stand for "circumference of circle," whenever it is evident that the circumference is intended for instance when circles are said to intersect

AXIOMS.

1 Things which are equal to the same thing are equal to one another

2. If equals be added to equals the wholes are equal

3 If equals be taken from equals the remainders are equal

4 If equals be added to unequals the wholes are unequal

5 If equals be taken from unequals the remainders are unequal

6 Things which are double of the same thing are equal to one another

7 Things which are halves of the same thing are equal to one another

8 Magnitudes which coincide with one another, that is, which may be made to fill exactly the same identical space, are equal to one another

9 The whole is greater than its part

10 Two straight lines cannot enclose a space

11. All right angles are equal to one another

12. See page 12.

Proposition 1 Problem

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line

It is required to describe an equilateral triangle on AB.

With A as centre and AB as radius, describe the circle BCD [*Post* 3

With B as centre and AB as radius, describe the circle ACE, intersecting the former circle at C [*Post* 3

Join CA and CB [*Post* 1

ABC *shall be an equilateral triangle*

Because the point A is the centre of the circle BCD,

therefore AC is equal to AB [*Def* 15

And because the point B is the centre of the circle ACE,

therefore BC is equal to BA [*Def* 15

But we have shewn, that AC is equal to AB,

and BC is equal to AB

But things which are equal to the same thing are equal to one another [*Ax* 1.

Therefore AC is equal to BC

Wherefore, CA, AB, BC are equal to one another, and *the triangle* ABC *is equilateral*, [*Def* 24.

and it is described on the given straight line AB

Which was to be done

QUESTIONS FOR EXAMINATION

1 What is a finite straight line?

2 What is an equilateral triangle?

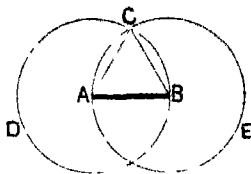
3 What is the datum in this proposition?

4 What is the quaesitum in this proposition?

5 What relation has AB to the circles described?

6 Into how many parts may this proposition be divided [*See Notes*

EXERCISES

1 If the extremities of AB be joined with the other point of intersection, shew that another equilateral triangle will be described on AB

2. On a given finite straight line to describe an isosceles triangle that shall have each of its sides double the base.

Proposition 2. Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal to BC.

From the point A to B draw the straight line AB; [*Post.* 1.
and upon AB describe the equilateral triangle DAB, [I. 1.
and produce the straight lines DA, DB to E and F respectively. [*Post.* 2.

From the centre B, at the distance BC, describe the circle CGH, meeting DF at G. [*Post.* 3.

From the centre D, at the distance DG, describe the circle KGL, meeting DE at L. [*Post.* 3.

The straight line AL *shall be equal to* BC.

Because the point B is the centre of the circle CGH,

therefore BC is equal to BG. [*Def.* 15.

And because the point D is the centre of the circle GKL,

therefore DL is equal to DG; [*Def.* 15.

and DA, DB, parts of them, are equal; [*Def.* 24.

therefore the remainder AL is equal to the remainder BG. [*Ax.* 3.

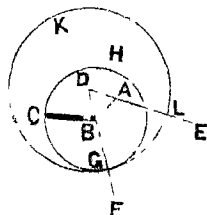
But it has been shewn that BC is equal to BG;

therefore AL and BC are each of them equal to BG.

But things which are equal to the same thing are equal to one another. [*Ax.* 1.

Therefore AL is equal to BC.

Wherefore, *from the given point* A *a straight line* AL *has been drawn equal to the given straight line* BC.

Which was to be done.

This Proposition may be omitted, for by the 3rd postulate we can place both the ends of a pair of compasses on the extremities of the given straight line and without changing the distance between the ends of the compasses we can transfer the compasses and place one end on the given point, and by postulate 1, we can join this point with the point where the other end falls, thus we can do what is required [See 'Notes' for the Use of Compasses]

EXERCISE

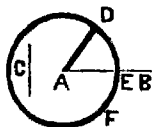
On the smaller of two given straight lines to describe an isosceles triangle that shall have each of its equal sides equal to the greater straight line

Proposition 3 Problem

From the greater of two given straight lines cut off a part equal to the less

Let AB and C be the two given straight lines, of which AB is the greater

It is required to cut off from AB, the greater, a part equal to C, the less

From the point A draw the straight line AD equal to C [I 2
and from the centre A, at the distance AD, describe the circle DEF meeting AB at E [*Post 3*
AE *shall be equal to* C

Because the point A is the centre of the circle DEF, therefore AE is equal AD [*Def* 15.
But C is equal to AD [*Cons*
Therefore AE and C are each of them equal to AD
Therefore AE is equal to C [*Ax*. 1

Wherefore, *from* AB *the greater of two given straight lines, a part* AE *has been cut off equal to* C *the less*

Which was to be done

NOTE For alternative proofs of Props 3, 5, 6, 8, see end of Book I

QUESTIONS FOR EXAMINATION

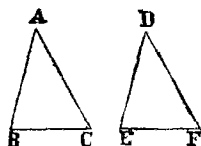
1. What postulate is used in the solution?

2 Shew how to produce the smaller of two given straight lines, so that with the part produced, it may be equal to the greater

Proposition 4. Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another then the two triangles shall be identically equal, that is, their bases or third sides shall be equal and their other angles shall be equal each to each, namely, those to which the equal sides are opposite

Let ABC DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, and the angle BAC equal to the angle EDF

The triangles ABC, DEF *shall be identically equal*

For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE

then the point B will coincide with the point E
because AB is equal to DE [*Hyp*

And AB coinciding with DE, AC must fall on DF,
because the angle BAC is equal to the angle EDF [*Hyp*

Therefore also the point C will coincide with the point F,
because AC is equal to DF [*Hyp*

But the point B was shewn to coincide with the point E,
therefore the base BC must coincide with the base EF,
because, B coinciding with E, and C with F if the base BC does not coincide with the base EF, two straight lines will enclose a space, which is impossible [*Ax* 10

Therefore the base BC coincides with the base EF,
and is equal to it. [*Ax* 8.

Wherefore the whole triangle ABC coincides with the whole
triangle DEF, and is equal to it [*Ax* 8

And the remaining angles of the one coincide with the remaining angles of the other, and are equal to them, namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Wherefore, *if two triangles &c*

Which was to be demonstrated

Questions for Examination

1 Mention the parts in the hypothesis of this proposition.

2 Mention the parts in the conclusion

3 What method is employed for the demonstration?

4 Shew how two equal straight lines may be made to coincide

5 Shew how two equal angles may be made to coincide

Exercises

1 If two squares have one side of the one equal to one side of the other, the squares are identically equal

2 If two equal triangles have one side and an adjacent angle in the one, equal to one side and an adjacent angle in the other, the remaining sides and angles shall be equal, each to each

3 If a perpendicular be erected at the middle point of a straight line, prove that every point on it is equidistant from the extremities of the straight line

4 If two adjacent sides of a quadrilateral be equal and if the diagonal bisect the angle between them shew that the other two sides are equal

5 The straight line which bisects the vertical angle of an isosceles triangle bisects the base perpendicularly

6 If on the sides of an equilateral triangle points be taken equally distant from the angles successively the straight lines joining these points will form a new equilateral triangle

Proposition 5. Theorem.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E:

Then the angle ABC *shall be equal to the angle* ACB, *and the angle* CBD *to the angle* BCE.

In BD take any point F,

and from AE the greater cut off AG equal to AF the less. [I. 3.

and join FC, GB.

Because AF is equal to AG, [*Constr.*

and AB to AC, [*Hyp.*

the two sides FA, AC are equal to the two sides GA, AB, each to each; and they contain the angle FAG which is common to the two triangles AFC, AGB; therefore the base FC is equal to the base BG, and the triangle AFC to the triangle AGB, and the remaining angles of the one to the remaining angles of the other, each to each, to which the equal sides are opposite, namely the angle ACF to the angle ABG, and the angle AFC to the angle AGB. [I. 4.

Because the whole AF is equal to the whole AG, of which the parts AB, AC are equal; [*Hyp.*

the remainder BF is equal to the remainder CG; [*Ax.* 3.

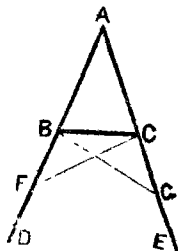
and FC was shewn to be equal to GB;

therefore the two sides BF, FC are equal to the two sides CG, GB, each to each;

and the angle BFC was shewn to be equal to the angle CGB;

therefore the triangles BFC, CGB are equal, and their other angles are equal, each to each, to which the equal sides are opposite, namely, the angle FBC to the angle GCB, and the angle BCF to the angle CBG. [I. 4.

And since it has been shewn that the whole angle ABG is equal to the whole angle ACF,

and that the parts of these, the angles CBG, BCF are also equal; therefore the remaining angle ABC is equal to the

remaining angle ACB which are the angles at the base of the triangle ABC [Ax 3

And it has also been proved that the angle FBC is equal to the angle GCB which are the angles on the other side of the base

Wherefore, *the angles at the base &c* Q E D

COR Hence every equilateral triangle is equiangular

EXERCISES

1 Any two opposite angles of a rhombus are equal

2 The diagonals of a rhombus bisect each other at right angles and divide the rhombus into four congruent triangles

3 The diagonals of a square bisect each other at right angles and divide the square into four congruent isosceles triangles

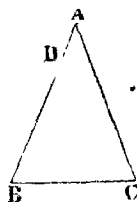
Proposition 6 Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles shall be equal to one another

Let ABC be a triangle having the angle ACB equal to the angle ABC

The side AB shall be equal to the side AC

For if AB be not equal to AC, one of them must be greater than the other

Let AB be the greater, and from it cut off BD equal to AC the less, and join DC [I 3

Then, because in the triangles DBC, ABC,

DB is equal to AC, [*Cons*

and BC is common to both triangles;

therefore the two sides DB, BC are equal to the two sides AC, CB, each to each,

and the angle DBC is equal to the angle ACB [*Hyp.*

therefore the triangle DBC is equal to the triangle ACB [I. 4

the less equal to the greater, which is absurd [*Ax* 9

Therefore, AB is not unequal to AC, that is, it is equal to it

Wherefore *if two angles &c* Q.E.D.

QUESTIONS FOR EXAMINATION

1 What is the hypothesis of this proposition, and what is the conclusion?

2 What is the converse of this proposition?

3 What is the obverse of prop 5?

4 What is the contrapositive to this proposition?

5 What is the method of proof used in this proposition?

6 What false assumption is made in the proof of this proposition?

EXERCISE

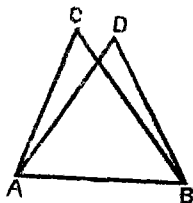
In figure, Euc, I 5 if BG, CF meet in H, shew that AH bisects the angle BAC

Proposition 7. Theorem.

Upon the same base and on the same side of it, there cannot be two triangles which have their sides terminated at one extremity of the base equal to one another and likewise those which are terminated at the other extremity equal to one another

If it be possible, on the same base AB and on the same side of it let there be two triangles ACB ADB having their sides CA, DA which are terminated at the extremity A of the base equal to one another, and likewise their sides CB, DB which are terminated at B equal to one another

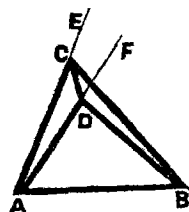
Join CD

FIRST *When the vertex of each of the triangles is without the other triangle*

Because AC is equal to AD, in the triangle ACD [*Hyp*
therefore the angle ACD is equal to the angle ADC [I 5
But the angle ACD is greater than the angle BCD, [*Ax* 9
therefore the angle ADC is also greater than the angle BCD
much more then is the angle BDC greater than the angle BCD.

Again, because the side BC is equal to BD [*Hyp*
therefore the angle BDC is equal to the angle BCD. [I 5
But it has been shewn to be greater, hence BDC is both equal to and greater than the angle BCD which is impossible

SECONDLY. *Let the vertex* D *of the triangle* ADB *be within the other triangle* ACB.

Produce AC to E and AD to F and join CD.

Then because AC is equal to AD, in the triangle ACD, [*Hyp.*
the angles ECD, FDC, on the other side of the base CD, are equal to one another. [I. 5.

But the angle ECD is greater than the angle BCD; [Ax. 9.
therefore the angle FDC is also greater than the angle BCD;
much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD, [*Hyp.*
therefore the angle BDC is equal to the angle BCD; [I. 5.
But it has been shewn to be greater; which is impossible.

THIRDLY. *The case in which the vertex of one triangle is on a side of the other*, needs no demonstration.

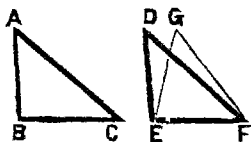
Wherefore, *upon the same base* &c. Q. E. D.

EXERCISE.

If two triangles on the same base and on the opposite sides of the base have their sides terminated at one extremity of the base equal, and likewise those which are terminated at the other extremity; the angle contained by the two sides of the one shall be equal to the angle contained by the two sides of the other.

Proposition 8. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle which is contained by the two sides, equal to them, of the other; and the triangles shall be equal in every respect.

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and also the base BC equal to the base EF.

Then the angle BAC *shall be equal to the angle* EDF, *and the triangles* ABC, DEF *shall be equal in every respect.*

For, if the triangle ABC be applied to the triangle DEF so that the point B may be on the point E and the straight line BC on the straight line EF, the point C shall coincide with the point F, because BC is equal to EF. [*Hyp.*

Therefore BC coinciding with EF, BA and CA will coincide with ED and FD.

For, if the base BC coincide with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG;
then upon the same base and on the same side of it there can be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those sides which are terminated at the other extremity.
But this is impossible. [I. 7.
Therefore if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, FD.

Therefore, likewise the angle BAC coincides with the angle EDF, and is equal to it. [*Ax.* 8.
By Prop. 4, the triangles may be proved to be equal in every respect.

Wherefore, *if two triangles have two sides* &c. Q. E. D.

EXERCISES

1 If two circles cut each other, the straight line joining their centres shall bisect at right angles the straight line joining the points of intersection.

* 2 If two isosceles triangles be upon the same base, but on opposite sides of it, the straight line joining the vertices will bisect each vertical angle.

3 The diagonals of a rhombus bisect each other at right angles and divide the rhombus into four congruent right-angled triangles.

Proposition 9. Problem

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle: it is required to bisect it.

Take any point D in AB; from AC cut off AE equal to AD; [I. 3.
join DE; and on DE, on the side remote from A, describe the equilateral triangle DEF. [I. 1.

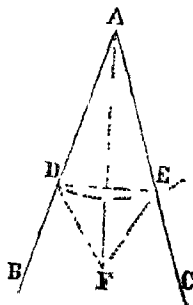
Join AF.

The straight line AF *shall bisect the angle* BAC.

Because AD is equal to AE, [*Cons.*
and AF is common to the two triangles DAF, EAF;

the two sides DA, AF are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF; [*Def.* 24.

therefore the angle DAF is equal to the angle EAF. [I. 8.

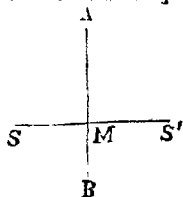
Wherefore, *the given rectilineal angle* BAC *is bisected by the straight line* AF. Q. E. F.

QUESTIONS FOR EXAMINATION

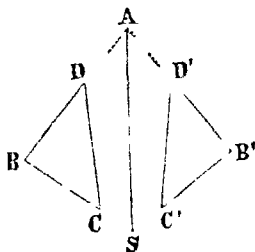
1 Why is the triangle described on the side remote from A?

2 What will be the consequence if BA be in the same straight line with AC?

3 Shew how by this proposition an angle may be divided into any number of equal parts denoted by the successive powers of the number 2

Def 1 *Two points are* ***symmetrical*** *in relation to a straight line, when the straight line bisects at right angles the join of the points and the straight line in relation to which any two points are symmetrical, is called the* ***axis of symmetry*** *of those points*

Thus, the points S S' are symmetrical in relation to AB when AB bisects SS' at right angles and AB is the axis of symmetry of the points S and S'

Def. 2 *If two figures in the same plane can be made to coincide by turning the one about a fixed straight line in the plane the fixed straight line is called the* ***axis of symmetry*** *of the two figures*

Let BCD and B'C'D' be two figures on the two sides of AS such that when BCD is turned about AS as axis, it coincides with B'C'D' then AS is the axis of symmetry of BCD and B'C'D'

EXERCISES

1 If two isosceles triangles be upon the same base whether on the same side or on opposite sides of the base, the straight line joining the vertices is the axis of symmetry in the quadrilateral formed

2 Every point in AF is equally distant from D and E

3 Draw an axis of symmetry in an isosceles triangle

4 If a quadrilateral have two adjacent sides equal and if the angle made by the sides be bisected by the diagonal drawn from that angle, shew that the diagonal is an axis of symmetry in the quadrilateral

5 The straight lines which bisect the angles at the base of an isosceles triangle form with the base a new isosceles triangle the axis of symmetry in which is the same as that in the original triangle.

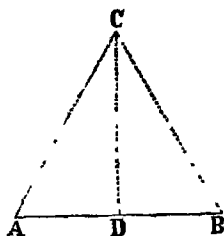
Proposition 10 Problem

To bisect a given finite straight line, that is, to divide it into two equal parts

Let AB be the given straight line *it is required to divide it into two equal parts*

Upon AB describe an equilateral triangle ABC [I 1

and bisect the angle ACB by the straight line CD, meeting AB at the point D [I 9

Then AB *shall be divided into two equal parts at the point* D

Because AC is equal to CB [*Def* 24

and CD is common to the two triangles ACD, BCD,

therefore in the two triangles ACD, BCD,
the sides AC, CD are respectively equal to the sides BC, CD,
and the angle ACD is equal to the angle BCD [*Cons*

therefore the base AD is equal to the base DB [I 4

Wherefore *the given straight line* AB *is divided into two equal parts at the point* D Q. E. F.

EXERCISES

1 To divide a straight line into 4, 8 16 &c equal parts

2 To bisect a straight line by the 3rd and 1st postulate

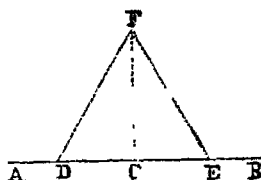
3 Every point equally distant from A and B is in CD or CD produced

4 The straight line drawn from the vertex of an isosceles triangle to the middle point of the base bisects the vertical angle.

Proposition 11. Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it: it is required to draw from the point C a straight line at right angles to AB.

Take any point D in AC, and make CE equal to CD. [I. 3.

On DE describe the equilateral triangle DFE, [I. 1.
and join CF.

CF *is the required straight line.*

Because DC is equal to EC, [*Cons.*
and CF is common to the two triangles DCF, ECF,
the two sides DC, CF are equal to the two sides EC, CF, each to each;
and the base DF is equal to the base EF; [*Def.* 24.
therefore the angle DCF is equal to the angle ECF; [I. 8.
and these two angles are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Def.* 11.
therefore each of the angles DCF, ECF is a right angle.

Wherefore, *from the given point* C *in the given straight line* AB, CF *has been drawn at right angles to* AB. Q.E.F.

EXERCISES.

1. From a given point in a straight line only one straight line can be drawn at right angles to the same.

2. To find a point in a given straight line so that it may be equidistant from two given points. When is the proposition impossible?

3. To find a point within a triangle so that it may be equidistant from the three angular points.

4. Given two points on opposite sides of a straight line, to find a point in the straight line so that the angle formed by joining the point with the given points shall be bisected by the given straight line.

5 From one end of a straight line, without producing it, draw a straight line making a right angle

6 Every point in FC is equally distant from DF and EF

Proposition 12. Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it

Let AB be the given straight line, which may be produced to any length both ways, and let C be the given point without it, it is required to draw from the point C a straight line perpendicular to AB

Take any point D on the other side of AB and with C as centre and CD as radius describe the circle EGF, meeting AB at F and G [*Post* 3

Bisect FG at H, [I 10

and join CH

CH *is the required straight line*

Join FC, CG

Because FH is equal to HG, [*Cons*

and HC is common to the two triangles FHC, GHC

the two sides FH, HC are equal to the two sides GH, HC, each to each,

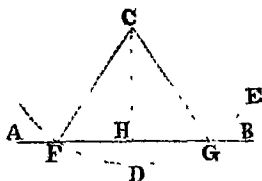
and the base CF is equal to the base CG, [*Def* 15

therefore the angle CHF is equal to the angle CHG, [I. 8

and they are adjacent angles

Therefore each of them is a right angle,

and CH is perpendicular to FG [*Def* 11.

Wherefore *from the given point* C *a perpendicular* CH *has been drawn to the given straight line* AB Q E F

EXERCISES

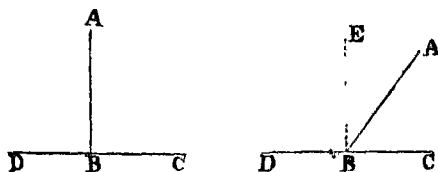
1 If the diagonals of a quadrilateral bisect each other at right angles, the figure is a lozenge

2 If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles

3 A point is equidistant from the angular points of a triangle, the straight lines joining this point with the middle points of the sides are at right angles to the sides

Proposition 13. Theorem.

The angles which one straight line makes with another straight line upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with the straight line CD, upon one side of it, the angles CBA, ABD, *then these are either two right angles, or are together equal to two right angles.*

For, if the angle CBA be equal to the angle ABD, each of them is a right angle [*Def.* 11

But, if the angle CBA be not equal to the angle ABD, from the point B draw BE at right angles to CD [I. 11

Now because the angle CBE is equal to the two angles CBA, ABE to each of these equals add the angle EBD therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD [*Ax.* 2.

Again because the angle DBA is equal to the two angles DBE, EBA to each of these equals add the angle ABC therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC [*Ax.* 2

But the angles CBE, EBD have been shewn to be equal to the same three angles

Therefore the angles CBE, EBD are equal to the angles DBA, ABC [*Ax.* 1

But CBE, EBD are two right angles, [*Ax.* 11

therefore DBA, ABC are together equal to two right angles

Wherefore, *the angles which &c.* Q. E. D.

Cor. 1 The sum of two supplementary angles is equal to two right angles

Cor. 2 If the equal sides of an isosceles triangle be produced, the angles on the other side of the base shall be equal to one another. (See second part Prop 5)

Cor. 3 Two straight lines cannot have a common segment

EXERCISES.

1 If an angle and its supplement be bisected, the bisecting lines are at right angles to each other

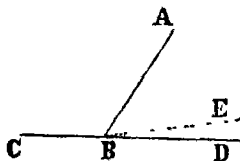
2 If the two exterior angles formed by producing a side of a triangle both ways are equal, shew that the triangle is isosceles.

Proposition 14. Theorem

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles

Then BD *shall be in the same straight line with* BC

For if BD be not in the same straight line with BC, let BE be in the same straight line with it

Then because the straight line AB meets the straight line CBE,

therefore the adjacent angles ABC, ABE are together equal to two right angles [I 13

But the angles ABC, ABD

are also together equal to two right angles [*Hyp*

Therefore the angles ABC, ABE

are equal to the angles ABC, ABD.

From each of these equals take away the common angle ABC.

Then the angle ABE is equal to the angle ABD, [*Ax* 3

the less to the greater, which is impossible

Therefore BE is not in the same straight line with CB.

Therefore BD is in the same straight line with BC

Wherefore, *if at a point in a straight line*, &c Q. E. D.

QUESTIONS

1 Of what proposition is this the converse?

2 If two straight lines cut each other, shew that the sum of all the angles at the point of intersection is equal to four right angles

EXERCISES.

1 The straight lines, joining the middle point of the diagonal of a rhombus with the opposite angles through which the diagonal is not drawn, are in one and the same straight line

2 The triangles ABC, DBE are so placed that AB, BE are in a straight line, if the angle ABC, be equal to the angle DBE, and if D, C be on opposite sides of ABE shew that the straight lines DB, BC are in one and the same straight line

Proposition 15. Theorem.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another at the point E

Then the angle AEC *shall be equal to the angle* DEB, *and the angle* CEB *to the angle* AED

Because the straight line AE makes with the straight line CD, the angles CEA, AED these angles are together equal to two right angles [I 13

Again, because the straight line DE makes with the straight line AB, the angles AED DEB, these angles also are together equal to two right angles [I 13.

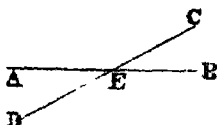
But the angles CEA, AED have been shewn to be together equal to two right angles

Therefore the angles CEA, AED are equal to the angles AED, DEB

From each of these equals take away the common angle AED,

then the angle CEA is equal to the angle DEB [Ax 3.

In the same manner it may be demonstrated that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines &c* Q E. D

COR 1 From this it is manifest, that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles

COR 2 And consequently, that all the angles made by any number of straight lines meeting at one point, are together equal to four right angles

EXERCISES.

1. State and prove the converse of Prop 15

2 If at a point in a straight line, two other straight lines meet on the opposite sides of it, and make the vertical or opposite angles equal, these two straight lines are in one and the same straight line

Proposition 16. Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let the side BC be produced to D then the *exterior angle* ACD *shall be greater than either of the interior opposite angles* CBA *or* BAC

Bisect AC in E, [I 10
join BE, and produce it to F, making EF equal to EB [I 3.
and join FC

Because in the triangles ABE, CFE,
AE is equal to EC, and BE to EF, [*Cons*
the two sides AE, EB are equal to the two sides CE, EF, each to each,
and the angle AEB is equal to the angle CEF, [I 15
therefore the angle BAE is equal to the angle ECF. [I 4

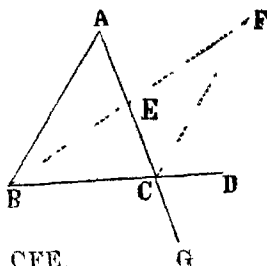
But the angle ECD or ACD is greater than the angle ECF, [*Ax* 9.

Therefore the angle ACD is greater than the angle BAE or BAC

Likewise, if the side BC be bisected, and the side AC be produced to G, it may be demonstrated that the angle BCG, that is, the angle ACD, is greater than the angle ABC [I 15.

Wherefore, *if one side &c* Q. E. D.

EXERCISES

1 From the same point there cannot be drawn more than two equal straight lines to meet a given straight line

2. Only one perpendicular can be drawn to a straight line from a given point without it

Proposition 17. Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle, *then any two of its angles are together less than two right angles*

Produce any side BC to D

Then because ACD is the exterior angle of the triangle ABC,

it is greater than the interior opposite angle ABC [I 16

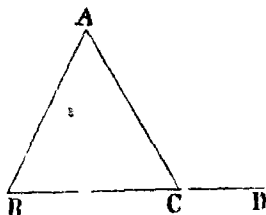
To each of these unequals add the angle ACB

Therefore the angles ACD, ACB are together greater than the angles ABC, ACB

But the angles ACD ACB are together equal to two right angles [I 13

Therefore the angles ABC, ACB are together less than two right angles

In like manner it may be demonstrated that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles

Wherefore, *any two angles* &c Q E D

EXERCISES

1 Prove this proposition without producing a side

2 The three interior angles of a triangle are together less than three right angles

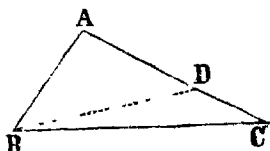
3 The two exterior angles of any triangle are together greater than two right angles, and the three exterior angles are together greater than three right angles

Proposition 18 Theorem.

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB :

Then the angle ABC *shall be greater than the angle* ACB

Because the side AC is greater than the side AB, make AD equal to AB, [I 3 and join BD

Then because in the triangle ABD, AD is equal to AB, therefore the angle ABD is equal to the angle ADB [I 5

But, because ADB is the exterior angle of the triangle BDC it is greater than the interior and opposite angle DCB [I 16

But the angle ADB has been proved equal to the angle ABD [I 5

Therefore the angle ABD is also greater than the angle ACB

Much more then is the angle ABC greater than the angle ACB [*Ax* 9

Wherefore, *the greater side* &c Q E D

EXERCISES.

1 Prove this proposition by producing the smaller side

2 With A as centre and AB the smaller side as radius describe a circle cutting BC or BC produced at D, and prove Proposition 18

3 In any triangle, the perpendicular drawn from the vertex to the opposite side, which is not less than either of the other sides, falls within the triangle

4 In a quadrilateral ABCD, the side AD is the greatest and BC the least; prove that the angle at B is greater than the angle at D, and the angle at C is greater than that at A

Proposition 19. Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB.

Then the side AC *is also greater than the side* AB.

For, if not, AC must be either equal to AB or less than AB.

But AC is not equal to AB;

For then the angle ABC would be equal to the angle ACB; [I. 5.

but it is not; [Hyp.

Therefore AC is not equal to AB.

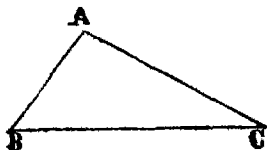
Neither is AC less than AB; for then the angle ABC would be less than the angle ACB; [I. 18.

But it is not; [Hyp.

Therefore AC is not less than AB.

And it has been shewn that AC is not equal to AB.

Therefore AC is greater than AB.

Wherefore, *the greater angle*, &c. Q. E. D.

EXERCISES.

1. Give a direct proof of this proposition.

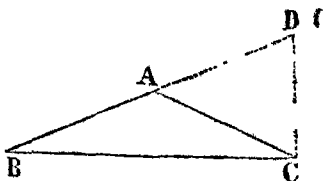
2. If BD and CD bisect the angles ABC and ACB respectively, and if AB be less than AC, then shew that BD is less than DC.

3. Of all the straight lines which can be drawn to a given straight line from a given point without it the perpendicular is the least; and of the rest, that which is nearer to the perpendicular is always less than one more remote; and there cannot be drawn more than two equal straight lines from the given point to the given straight line.

Proposition 20. Theorem.

Any two sides of a triangle are together greater than the third side

Let ABC be a triangle then *any two sides of it are together greater than the third side*

Produce the side BA to the point D, making AD equal to AC, [I 3
and join DC

Then because AD is equal to AC, [*Cons.*
therefore the angle ACD is equal to the angle ADC [I 5
But the angle BCD is greater than the angle ACD [*Ax* 9.
Therefore the angle BCD is also greater than the angle ADC

Therefore the side BD is greater than the side BC [I 19

But BD is equal to BA and AC

Therefore BA, AC are together greater than BC

In the same manner it may be demonstrated that AB, BC are greater than AC and BC CA greater than AB

Wherefore, *any two sides* &c Q E D

EXERCISES

1 Any side of a triangle is greater than the difference between the other two sides

Def *The sum of the sides of a rectilineal figure is called its* ***perimeter***

2 The sum of the distances of any point within a triangle from the angular points is greater than half its perimeter

3 The perimeter of a quadrilateral is greater than the sum of its diagonals

Def *A straight line drawn from any angle of a triangle to the middle point of the opposite side is called a* ***median*** *of the triangle*

4 The sum of the three medians of a triangle is less than its perimeter

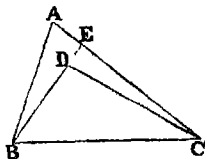
5 The sum of two sides of a triangle is greater than double the straight line drawn from the vertex to the middle point of the base

6 In a given straight line to find a point such that the sum of two straight lines drawn to it from two given points without the given line and on the same side of it, shall be less than the sum of any two lines drawn from the same points, and terminated at any other point in the same line

Proposition 21 Theorem

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle but shall contain a greater angle

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle:

Then BD, DC *shall be less than the two sides* BA, AC, *but shall contain an angle* BDC *greater than the angle* BAC

Produce BD to meet the side AC at E

In the triangle ABE, the sum of the sides BA, AE is greater than the side BE [I 20

To each of these unequals add CE,
then the sum of BA, AC is greater than the sum of BE EC

Again, because the two sides CE ED of the triangle CED are together greater than the third side CD, [I. 20
to each of these unequals add DB

Therefore the sum of CE, EB is greater than the sum of CD, DB [Ax 4.

But it has been shewn that the sum of BA AC is greater than BE, EC, much more then is the sum of BA, AC greater than the sum of BD DC

The exterior angle BDC of the triangle CDE is greater than the angle CED. [I 16

For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE

Much more then is the angle BDC greater than the angle BAC

Wherefore, *if from the ends,* &c Q E D

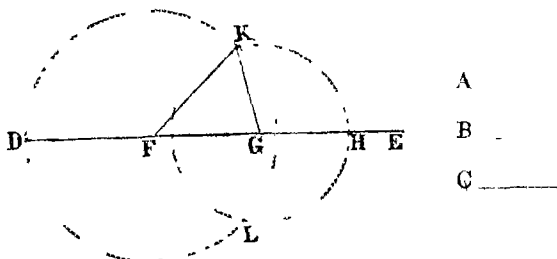
EXERCISES

1 Without producing BD, join AD and produce it towards the base, and prove that the angle BDC is greater than the angle BAC

2 If from a point within a triangle, straight lines be drawn to the vertices of the three angles, these three straight lines taken together shall be less than the sum of the three sides, but greater than half that sum

Proposition 22. Problem.

To make a triangle of which the sides shall be equal to three given straight lines, any two of which together being greater than the third

Let A, B, C be the three given straight lines, of which the sum of any two whatever are greater than the third it is required to make a triangle of which the sides shall be equal to A, B, C, each to each

Take a straight line DE and in it take any point F and make FG equal to B [I 3.
With F as centre and A as radius, describe the circle DKL.
With G as centre and C as radius describe the circle HLK intersecting the former circle at K Join KF, KG
Then KFG *is the required triangle*
Then because KF is equal to A, [*Cons*
FG equal to B, and [*Cons*
GK equal to C [*Cons*.
therefore the three straight lines KF, FG, GK, are respectively equal to the three straight lines A, B, C

Wherefore, *the triangle* KFG *has its three sides* KF, FG, GK, *equal to three given straight lines* A, B, C Q E F

QUESTIONS FOR EXAMINATION.

1. Shew that Prop 1 is a particular case of Prop. 22.

2. What is the reason for the condition that the sum of any two of the given straight lines must be greater than the third?

3. Shew when the circles will not intersect.

4. When will the circles meet but not intersect?

EXERCISES.

1. To construct a triangle equal to a given triangle.
2. To describe a rectilineal figure equal to a given rectilineal figure.

Proposition 23. Problem.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle: it is required, at the given point A in the given straight line AB, to make an angle that shall be equal to the given rectilineal angle DCE.

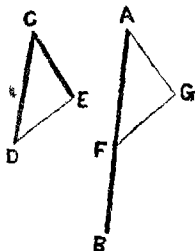
In CD, CE take any points D, E, and join DE.

On AB, make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC respectively, so that AF shall be equal to CD, AG to CE, and FG to DE. [I. 22.

Then the angle FAG *shall be equal to the angle* DCE.

Because, FA, AG are equal to DC, CE, each to each, and the base FG equal to the base DE; [*Cons.*

therefore the angle FAG is equal to the angle DCE. [I. 8.

Wherefore, *at the given point* A *in the given straight line* AB, *the angle* FAG *is made equal to the given rectilineal angle* DCE. Q. E. F.

EXERCISES.

1. Given two sides and the angle between them, construct the triangle.

2. Given the base and the angles at the base, construct the triangle.

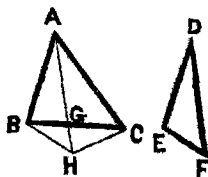
3. Given the base, an angle at the base and the sum of the other two sides, construct the triangle.

4. Given the base, an angle at the base and the difference of the other two sides, construct the triangle.

Proposition 24. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other the base of that which has the greater angle shall be greater than the base of the other.

Let ABC DEF be two triangles, in which the side AB is equal to the side DE, and the side AC to the side DF, but the angle BAC is greater than the angle EDF

then *the base* BC *shall be greater than the base* EF

Of the two sides AB, AC let AB be the side which is not greater than the other At the point A in the straight line BA and on the same side of it as AC, make the angle BAG equal to the angle EDF, AG meeting BC at G [I. 23.
Because AB is not greater than AC therefore the angle ACB is not greater than the angle ABC [I 5, 18.
But the angle AGC is greater than the angle ABC [I. 16
Therefore the angle AGC is greater than the angle ACB
Therefore the side AC is greater than the side AG [I 19.

Produce AG to H making AH equal to AC or DF
Join CH, BH

Because in the triangles ABH, DEF,
the side AB is equal to DE, [*Hyp*
and AH to DF, [*Cons.*
and the angle BAH is equal to the angle EDF, [*Cons*
therefore the base BH is equal to the base EF [I 4

And because AH is equal to AC in the triangle AHC, [*Cons*
therefore the angle ACH is equal to the angle AHC. [I 5
But the angle BHC is greater than the angle AHC [*Ax* 9.
Therefore the angle BHC is greater than the angle ACH
Much more then is the angle BHC greater than the angle BCH, [*Ax* 9.
therefore the side BC is greater than the side BH [I. 19.

But BH was proved equal to EF,
therefore BC is greater than EF.

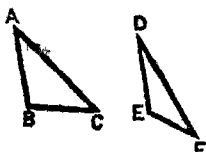
Wherefore, *if two triangles, &c.* Q. E. D.

EXERCISE

Prove that the angle ACB is greater than the angle AHB.

Proposition 25 Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB equal to DE, and AC to DF, but the base BC greater than the base EF, *the angle* BAC *shall be greater than the angle* EDF

For, if the angle BAC be not greater than the angle EDF, it must be either equal to the angle EDF or less than the angle EDF But the angle BAC is not equal to the angle EDF for then the base BC would be equal to the base EF [I 4

which it is not [*Hyp*

Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF, [I 24

which it is not. [*Hyp*

Hence the angle BAC is neither equal to, nor less than the angle EDF,

therefore the angle BAC is greater than the angle EDF

Wherefore, *if two triangles* &c Q E D

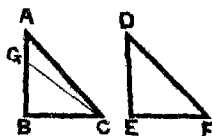
EXERCISE

1. Give a direct proof of Prop. 25

Proposition 26 Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of the one equal to a side of the other, viz., either the sides adjacent to the equal angles, or the sides opposite to one of the equal angles in each; then the triangles are identically equal.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, namely, ABC to DEF, and BCA to EFD, and also one side equal to one side.

First, let those sides be equal which are adjacent to the equal angles, namely, BC to EF then *the triangles are equal in every respect*

For if AB be not equal to DE, make BG equal to ED, [I 3. and join GC

Then because in the two triangles GBC, DEF,
GB is equal to DE, [*Cons*
and BC to EF, [*Hyp*
and the angle GBC is equal to the angle DEF, [*Hyp*
therefore the angle GCB is equal to the angle DFE [I 4.
But the angle DFE is equal to the angle ACB [*Hyp*.
Therefore the angle GCB is equal to the angle ACB, [*Ax* 1.
the less to the greater; which is impossible
Therefore AB is not unequal to DE, that is, it is equal to it,
therefore, in the two triangles ABC, DEF the sides AB, BC are respectively equal to the sides DE, EF,
and the angle ABC is equal to the angle DEF, [*Hyp*
therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF.* [I. 4.

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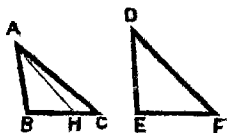
* This part of the proposition may be proved by superposition For if the triangle DEF be applied to the triangle ABC so that E falls on B and EF on BC, then EF will coincide with BC and the point F with C, for BC is equal to EF Because the angles DEF and DFE are respectively equal to the angles ABC and ACB, therefore ED, FD must fall on BA and CA respectively and will coincide with them Therefore the triangles ABC and DEF are congruent

Secondly, let the sides which are opposite to one of the equal angles in each triangle be equal to one another, namely, AB to DE, then in this case likewise, *the triangles are equal in every respect*

If BC be not equal to EF, make BH equal to EF, [I 3 and join AH

Then because in the two triangles ABH and DEF,

AB is equal to DE, [*Hyp.*
BH is equal to EF, [*Cons*
and the angle ABH is equal to the angle DEF, [*Hyp*

therefore the angle BHA is equal to the angle EFD [I 4.

But the angle EFD is equal to the angle BCA [*Hyp*

Therefore the angle BHA is equal to the angle BCA, [*Ax* 1 that is, the exterior angle BHA of the triangle AHC is equal to the interior and opposite angle HCA,

which is impossible [I 16

Therefore BC is not unequal to EF, that is, it is equal to it

Therefore, in the two triangles ABC, DEF,

the sides AB, BC of the one are respectively equal to the sides DE, EF of the other,
and the angle ABC is equal to the angle DEF [*Hyp*

therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF [I 4

Wherefore, *if two triangles &c* Q E D

EXERCISES

1 If the hypotenuse and one side of a right angled triangle be equal to the hypotenuse and one side of another right angled triangle, the triangles are congruent

2 If the hypotenuse and one of the acute angles of a right-angled triangle be equal to the hypotenuse and one of the acute angles of another, the triangles are congruent

3 If the perpendicular on the base from the vertex of a triangle bisect the vertical angle, the triangle is isosceles

4. In an isosceles triangle, the perpendicular from the vertex on the base bisects the vertical angle

5 Two straight lines make an angle between them, draw through a given point a straight line which will make equal angles with the given straight lines

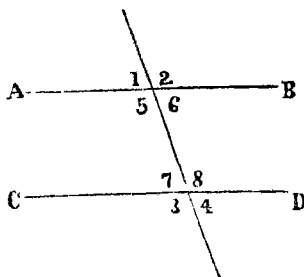
6 Given three points not in a straight line through one of them draw a straight line so that perpendiculars on it from the other two points on opposite sides of the line may be equal.

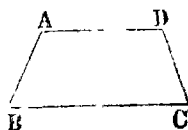
PARALLEL STRAIGHT LINES

DEF 37 **Parallel straight lines** *are such as are in the same plane and which being produced ever so far both ways do not meet*

DEF 38 *If a straight line fall on two other straight lines, the angles which are thus formed have received special names*

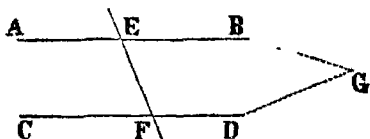
DEF 39 In the annexed figure, 1, 2, 3, 4 are called **exterior angles**, 5, 6 7, 8 are called **interior angles**, 5, 8 and 6, 7 are called **alternate angles**,

1, 7, 2, 8, 3, 5, 4, 6 are sometimes called **corresponding angles**

DEF 40 *A quadrilateral which has its opposite sides parallel is called a* **parallelogram.**

DEF 41 *A quadrilateral which has one pair of opposite sides parallel is called a* **trapezoid.**

DEF. 42 *If each pair of opposite sides in a quadrilateral be produced to meet, the straight line joining the points of intersection is called its* **third diagonal,** *and the figure so formed is called a* **complete quadrilateral.**

Proposition 27 Theorem.

If a straight line falling on two other straight lines, make the alternate angles equal to one another the two straight lines shall be parallel

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another *then* AB *shall be parallel to* CD

For, if AB be not parallel to CD, then AB and CD, being produced, will meet either towards B, D or towards A C Let them be produced and meet, if possible, towards B, D at the point G Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG . [I. 16

But the angle AEF is also equal to the angle EFG , [*Hyp*

which is impossible

Therefore AB and CD, being produced, do not meet towards B, D In like manner, it may be demonstrated that they do not meet towards A, C

But those straight lines in the same plane which being produced ever so far both ways do not meet are parallel to one another , therefore AB is parallel to CD [*Def.* 37

Wherefore, *if a straight line* &c Q E D

QUESTIONS FOR EXAMINATION

1 Shew that two perpendiculars to a straight line are parallel

2 If the angles of a quadrilateral be all right angles shew that it is a parallelogram

EXERCISES

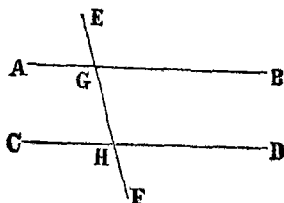
1 If the opposite sides of a quadrilateral be equal, it is a parallelogram

2 The bisectors of any pair of alternate angles are parallel

3 A rhombus and a rhomboid are both parallelograms.

Proposition 28. Theorem.

If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line; or make the interior angles on the same side, together equal to two right angles; the two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side of the line EF; or make the two interior angles on the same side BGH, GHD together equal to two right angles; then AB *shall be parallel to* CD.

Because the angle EGB is equal to the angle GHD, [*Hyp.*
and the angle EGB is also equal to the angle AGH, [I. 15.
therefore the angle AGH is equal to the angle GHD; [*Ax.* 1.
and they are alternate angles;
therefore AB is parallel to CD. [I. 27.
Again, because the angles BGH, GHD are together equal to two right angles, [*Hyp.*
and that the angles AGH, BGH are also together equal to two right angles, [I. 13.
therefore the angles AGH, BGH are together equal to the angles BGH, GHD.
Take away from these equals the common angle BGH;
therefore the remaining angle AGH is equal to the remaining angle GHD; [*Ax.* 3.
and they are alternate angles, therefore AB is parallel to CD. [I. 27.

Wherefore, *if a straight line, &c.* Q. E. D.

12th Axiom.

If a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

QUESTION FOR EXAMINATION.

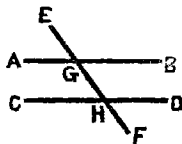
Enunciate the two separate propositions of which Prop. 28 is made up

EXERCISE

In a quadrilateral ABCD, if the angle at B be equal to the angle at D, and the exterior angle at D equal to the angle at A, shew that the quadrilateral is a parallelogram

Proposition 29 Theorem

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another and the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles on the same side together equal to two right angles

Let the straight line EF fall on the two parallel straight lines AB, CD, *then the alternate angles* AGH, GHD *shall be equal to one another, and the exterior angle* EGB *shall be equal to the interior and opposite angle* GHD *on the same side of the line* EF, *and the two interior angles* BGH, GHD *on the same side of* EF *shall be together equal to two right angles*

First For if the alternate angle AGH be not equal to the alternate angle GHD, one of them must be greater than the other, let the angle AGH be the greater

To each of these unequals add the angle BGH,

therefore the angles AGH, BGH are together greater than the angles BGH, GHD [*Ax* 4

But the angles AGH, BGH are together equal to two right angles, [I 13

therefore the angles BGH, GHD are together less than two right angles

Therefore the straight lines AB, CD, if produced far enough, will meet towards B, D [*Ax* 12

But they never meet, since they are parallel. [*Hyp*

Therefore the angle AGH is not unequal to the angle GHD, that is, the angle AGH is equal to the angle GHD

Secondly Because the angle AGH is equal to the angle EGB, [I 15

and the angle AGH is equal to the angle GHD,

therefore the exterior angle EGB is equal to the interior and opposite angle GHD on the same side of the line EF [*Ax* 1

Thirdly. Because the angle EGB is equal to the angle GHD

Add to each of these the angle BGH

Therefore the angles EGB, BGH are together equal to the angles BGH, GHD.

But the angles EGB, BGH are together equal to two right angles [I 13

Therefore also the two interior angles BGH, GHD on the same side of the line EF are together equal to two right angles

Wherefore, *if a straight line*, &c Q E D

QUESTIONS FOR EXAMINATION

1 Enunciate the three separate propositions of which Prop 29 is made up

2. Enunciate the three converse cases of Prop 29

EXERCISES

1 Two different straight lines cannot be drawn through the same point parallel to a given straight line

2 Two angles which have their arms parallel are equal or supplementary

3 The straight line which is perpendicular to one of two parallel straight lines is also perpendicular to the other

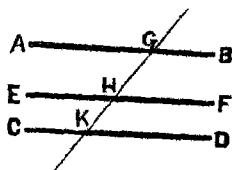
4 The parts of all perpendiculars to two parallel straight lines intercepted between them are equal

5 If a straight line which meets two parallel straight lines be bisected, any straight line drawn through the point of bisection and terminated by the parallel lines is also bisected at that point

Proposition 30. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.

Let the straight lines AB, CD be each of them parallel to EF; then AB *shall be parallel to* CD.

Let the straight line GHK cut AB, EF, CD.

Then because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the alternate angle GHF. [I. 29.

Again, because GK cuts the parallel straight lines EF, CD, therefore the exterior angle GHF is equal to the interior and opposite angle GKD. [I. 29.

But it has been shewn that the angle AGK is equal to the angle GHF.

Therefore the angle AGK is equal to the angle GKD; [Ax. 1.

and these are alternate angles;

therefore AB is parallel to CD. [I. 27.

Wherefore, *straight lines &c.* Q. E. D.

EXERCISE.

If two straight lines which pass through a point are parallel to a third straight line, they are in the same straight line.

Proposition 31. Problem.

To draw a straight line through a given point parallel to a given straight line.

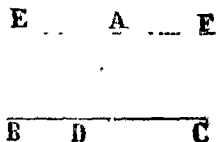
Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A parallel to the straight line BC.

In the straight line BC take any point D, and join AD; at the point A in the straight line AD, make the angle DAE equal to the angle ADC on the opposite side of AD; [I. 23.

E A F

B D C

and produce the straight line EA to F;

then EF *shall be parallel to* BC.

Because the straight line AD meets the two straight lines EF, BC, and makes the alternate angles EAD, ADC equal to one another, therefore EF is parallel to BC. [I. 27.

Wherefore, *through the given point* A, *has been drawn a straight line* EAF *parallel to the given straight line* BC Q. E. F.

EXERCISES.

1. Through a given point to draw a straight line to meet another straight line and make an angle equal to a given rectilineal angle

2 Of all triangles having the same vertical angle and whose bases pass through the same point, the least is that whose base is bisected at that point.

3 To trisect a finite straight line.

4 Through a given point draw a straight line cutting two parallel straight lines so that the part of it intercepted by the parallel lines may be equal to a given line Shew when the construction fails

5 If the straight line which bisects an exterior angle at the vertex of a triangle be parallel to the base, the triangle is isosceles

6 If from any point in the base of an isosceles triangle a straight line be drawn at right angles to the base cutting a side and meeting the other side produced, shew that the triangle thus formed is isosceles

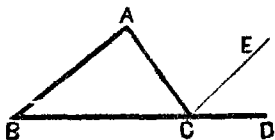
7 Through two points in two parallel straight lines draw two lines so as to make a rhombus

8 Draw a straight line parallel to the base of an isosceles triangle so as to form a trapezoid, the three smaller sides of which will be equal to one another

9 Draw a straight line parallel to the base of a triangle so as to form a trapezoid, the smaller parallel side of which will be equal to the sum of the intercepts on the sides from the ends of the base

Proposition 32. Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D; then *the exterior angle* ACD *shall be equal to the two interior and opposite angles* CAB, ABC, *and the three interior angles* ABC, BCA, CAB *shall be equal to two right angles.*

Through the point C draw CE parallel to BA. [I. 31.

Then because CE is parallel to BA, and AC meets them, therefore the angle ACE is equal to the alternate angle BAC. [I. 29.

Again, because CE is parallel to BA, and BD falls upon them, therefore the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29.

But the angle ACE has been shewn to be equal to the angle BAC;

therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC.

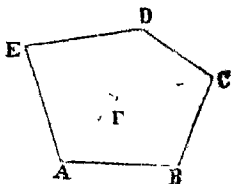
Again, because the angle ACD is equal to the two angles ABC, BAC, to each of these equals add the angle ACB; therefore the angles ACD and ACB are together equal to the three angles ABC, BCA, CAB. [*Ax.* 2.

But the angles ACD, ACB are together equal to two right angles, [I. 13.

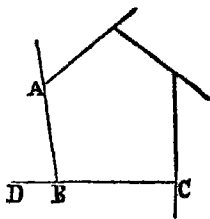
therefore also the angles ABC, BCA, CAB are together equal to two right angles. [*Ax.* 1.

Wherefore, *if a side of any triangle be produced*, &c. Q.E.D.

COR. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For, any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides by drawing straight lines from a point F within the figure to each of the angles Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides, therefore all the angles of these triangles are together equal to twice as many right angles as the figure has sides But the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point F, and the angles at the point F, which is the common vertex of all the triangles are equal to four right angles [I 15, *Cor* 2. Therefore the same angles of these triangles are equal to the interior angles of the figure together with four right angles

But it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides Therefore all the interior angles of the figure together with four right angles are equal to twice as many right angles as the figure has sides

COR 2 All the exterior angles of any rectilineal figure made by producing the sides successively in the same direction are together equal to four right angles

Since every interior angle ABC with its adjacent exterior angle ABD is equal to two right angles, [I 13 therefore all the interior angles, together with the exterior angles are equal to twice as many right angles as the figure has sides, but it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides, therefore all the interior angles together with all the exterior angles are equal to all the interior angles and four right angles [*Ax* 1 Take away from these equals all the interior angles Therefore all the exterior angles of the figure are equal to four right angles [*Ax* 3

COR 3 In a right-angled isosceles triangle each acute angle is half a right angle

COR 4 Each angle of an equilateral triangle is two-thirds of a right angle

COR 5. If two angles of one triangle be equal to two angles of another triangle the remaining angle of the first triangle is equal to the remaining angle of the second

COR. 6 If one angle of a triangle be equal to the sum of the other two, the greatest angle is a right angle.

COR 7. The angles at the base of an isosceles triangle are acute

EXERCISES.

1. The sum of the angles of a quadrilateral is equal to four right angles

2 In a right angled triangle, the hypotenuse is double of the median to the hypotenuse

3. Trisect a right angle

4 Each of the angles made with the base by the perpendiculars to the equal sides of an isosceles triangle drawn from the ends of the base is equal to half the vertical angle

5 If one of the acute angles of a right-angled triangle be double of the other, prove that the hypotenuse is double of the smaller side

6 If the opposite angles of a quadrilateral figure be equal to one another, the figure is a parallelogram

7 If a straight line be drawn from one of the angles of a triangle making the exterior angle equal to the sum of the two interior and opposite angles, it is in the same straight line with the adjacent side

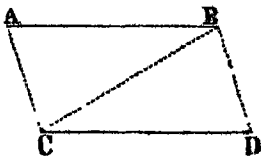
8 If two straight lines which cut one another be respectively perpendicular to two others which cut one another, the angles contained by the first two are respectively equal to the angles contained by the others

9 Any angle of a triangle is acute, right, or obtuse, accordingly as the straight line drawn from its vertex bisecting the opposite side, is greater than equal to, or less than half of that side

10 If the sides of an equilateral and equiangular hexagon be produced till they meet, the angles formed at the points of meeting are together equal to four right angles

Proposition 33. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are themselves equal and parallel.

Let AB and CD be two equal and parallel straight lines and let them be joined towards the same parts by the straight lines AC and BD, *then* AC *and* BD *shall be equal and parallel* Join BC

Then because AB is parallel to CD, [*Hyp*
and BC meets them,

therefore the alternate angles ABC, DCB are equal. [I 29.
And because AB is equal to CD, [*Hyp*
and BC is common to the two triangles ABC, DCB,

the two sides AB, BC are equal to the two sides DC, CB, each to each, and the angle ABC has been proved to be equal to the angle DCB,

therefore the base AC is equal to the base DB,
and the angle ACB is equal to the angle DBC [I. 4.
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, DBC equal to one another, therefore AC is parallel to BD. [I. 27
And it has been shewn to be equal to it

Wherefore, *the straight lines* &c Q. E. D

EXERCISES

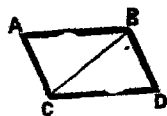
1 The straight line joining the points of bisection of any two sides of a triangle is parallel to the remaining side and also half of it

2 The straight lines joining successively the middle points of the sides of a quadrilateral form a parallelogram.

Proposition 34 Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram, that is, divides it into two equal parts

Let ABDC be a parallelogram, of which BC is a diagonal, *then the opposite sides and angles of the figure shall be equal to one another, and the diagonal* BC *shall bisect it*

Because AB is parallel to CD, and BC meets them, therefore the alternate angles ABC, DCB are equal to one another [I 29

And because AC is parallel to BD, and BC meets them therefore the alternate angles ACB, DBC are equal to one another [I 29.

Therefore the two triangles ABC, DCB have the two angles ABC BCA in the one, equal to the two angles DCB, CBD in the other, each to each, and a side BC, which is adjacent to the equal angles, common to the two triangles

therefore the two triangles are congruent [I 26

the side AB is equal to the side CD, and the side AC equal to the side BD, and the angle BAC equal to the angle CDB

And because the angle ABC is equal to the angle DCB, and the angle CBD to the angle ACB,

therefore the whole angle ABD is equal to the whole angle ACD [Ax 2

Therefore the opposite sides and angles of a parallelogram are equal to one another

Also the diagonal BC bisects the parallelogram;

for the triangle ABC has been shewn to be equal to the triangle DCB.

Wherefore, *the opposite sides* &c. Q. E. D.

EXERCISES

1 The diagonals of a parallelogram bisect each other

2 The quadrilateral whose diagonals bisect each other is a parallelogram

3 The diagonals of a right angled parallelogram are equal.

4 The parallelogram whose diagonals cut each other at right angles is a lozenge

5 If a straight line be drawn through the middle point of one side of a triangle, parallel to the base, it will bisect the other side

6 The straight line drawn from the middle point of the hypotenuse of a right angled triangle to the right angle is equal to half the hypotenuse

Def *The* ***orthogonal projection*** *of one straight line on another straight line is the portion of the latter intercepted between perpendiculars let fall on it from the extremities of the former*

Def *The portion of a line intercepted between two other lines is called an* ***intercept***

7 The straight lines that are equal and parallel have equal projections on any other straight line

8 Equal straight lines which have equal projections on another straight line are parallel

9 If one pair of parallel straight lines be parallel to another pair, and if the intercepts made by each pair on a straight line that cuts them are equal then the intercepts on any other straight line that cuts them are also equal

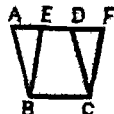
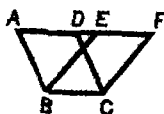
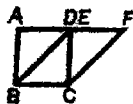
10 From a given point draw a straight line to cut two parallel straight lines so that its part intercepted between them may be equal to a given straight line not less than the distance between the parallel lines

11 Divide a straight line into any number of equal parts

12 Bisect a parallelogram by a straight line drawn through any given point

Proposition 35 Theorem.

Parallelograms on the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC, *then the parallelogram* ABCD *shall be equal to the parallelogram* EBCF.

Because DC is parallel to AB and AF cuts them,
the angle EAB is equal to the angle FDC, [I. 29.
and because FC is parallel to EB and AF cuts them,
the angle AEB is equal to the angle DFC, [I 29
therefore in the triangles EAB and FDC
the angles EAB and AEB are respectively equal to the angles FDC and DFC,
and the side AB is equal to the side DC, [I 34
therefore the triangle EAB is equal to the triangle FDC [I 26
Take away the triangle FDC from the trapezium ABCF, and from the same trapezium take away the triangle EAB, then the remainders are equal that is,
the parallelogram ABCD is equal to the parallelogram EBCF

Wherefore, *parallelograms on the same base*, &c Q E D

EXERCISES.

1 Equal parallelograms upon the same base and on the same side of it are between the same parallels

2 The parallelograms described on any two sides of a triangle are together equal to the parallelogram described on the base, having its side equal and parallel to the straight line drawn from the point of intersection of the exterior sides of the former, to the vertex of the triangle

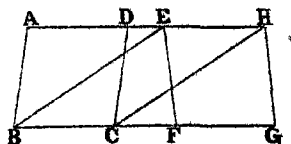
3 If the base of a parallelogram be equal to half the sum of the two parallel sides of a trapezoid, between the same parallels, the parallelogram is equal to the trapezoid.

Proposition 36. Theorem.

Parallelograms upon equal bases, and between the same parallels are equal to one another

Let ABCD, EFGH be two parallelograms on equal bases BC, FG, and between the same parallels AH, BG · *then the parallelogram* ABCD *shall be equal to the parallelogram* EFGH

Join BE, CH

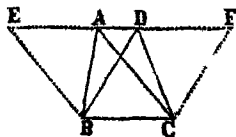
Because BC is equal to FG, [*Hyp*
and FG to EH, [I 34
therefore BC is equal to EH, [*Ax* 1
and these lines are parallel, [*Hyp*
and joined towards the same parts by the straight lines BE, CH.
Therefore BE, CH are both equal and parallel [I 33.
Therefore EBCH is a parallelogram, [*Def* 40.
and is equal to ABCD, because they are upon the same base BC, and between the same parallels, BC, AH [I 35
For the same reason, the parallelogram EFGH is equal to the same parallelogram EBCH
Therefore the parallelogram ABCD is equal to the parallelogram EFGH [*Ax* 1.

Wherefore, *parallelograms* &c Q. E. D.

Proposition 37. Theorem.

Triangles on the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC

then the triangle ABC *shall be equal to the triangle* DBC

Produce AD both ways to the points E, F, [*Post.* 2.

through B draw BE parallel to CA, and through C draw CF parallel to BD [I. 31.

Then EBCA, DBCF are parallelograms; [*Def.* 40.

and EBCA is equal to DBCF,

because they are upon the same base BC, and between the same parallels BC, EF. [I. 35.

And the triangle ABC is half of the parallelogram EBCA,

because the diagonal AB bisects the parallelogram; [I. 34.

and the triangle DBC is half of the parallelogram DBCF,

because the diagonal DC bisects the parallelogram [I. 34.

But the halves of equal things are equal [*Ax.* 7.

Therefore the triangles ABC and DBC are equal.

Wherefore, *triangles on the same base* &c. Q. E. D.

EXERCISES

1. To construct a triangle which shall be equal to a given trapezium, and shall have one side equal to a side of the trapezium.

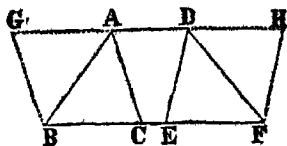
2. On the base of a given triangle, construct another triangle equal in area to the first, and having its vertex on a given straight line.

3. If two triangles equal in area be on the same base but on opposite sides, the straight line joining their vertices is bisected by the base.

4. Given the area of a triangle and the base, find the locus of the vertex.

Proposition 38. Theorem.

Triangles upon equal bases, and between the same parallels, are equal to one another

Let the two triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD: *then the triangle* ABC *shall be equal to the triangle* DEF.

Produce AD both ways to the points G, H, through B draw BG parallel to CA, and through F draw FH parallel to ED. [I 31

Then each of the figures GBCA, DEFH is a parallelogram [*Def* 40

And they are equal to one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH [I 36

And the triangle ABC is half of the parallelogram GBCA, because the diagonal AB bisects the parallelogram, [I 34

and the triangle DEF is half of the parallelogram DEFH, because the diagonal DF bisects the parallelogram. [I 34

But the halves of equal things are equal. [*Ax* 7

Therefore the triangle ABC is equal to the triangle DEF.

Wherefore, *triangles* &c Q. E. D.

EXERCISES

1 Every median bisects the triangle

2. To divide a triangle into any number of equal parts by drawing straight lines from the vertex to the base

3 The straight line, drawn from the vertex of a triangle to the middle point of the base, bisects every straight line parallel to the base and terminated by the other sides of the triangle.

4 The diagonal of a quadrilateral which bisects the other diagonal, bisects the quadrilateral.

5 If two sides of one triangle be equal to two sides of another triangle and the contained angle of the first is supplemental to the contained angle of the second, the triangles are equal in area

6 If a point in the median to the base be joined to the ends of the base, the triangle formed by the joins and the base is bisected by the median.

7 Any line drawn through the middle point of a diagonal of a parallelogram bisects the parallelogram.

8 To bisect a triangle by drawing a straight line through any point in one of its sides

9 Straight lines are drawn from G, C, to any point K in AB; shew that the triangle AKG is equal to the triangle AKC

10 A triangle is four times the triangle cut off from it by the straight line joining the middle points of any two sides

11 If two equal triangles on equal bases in a straight line stand on the same side of it, the intercepts on any line parallel to the base are equal

12 To trisect a triangle by drawing two straight lines from a point in one of the sides

Proposition 39 Theorem

Equal triangles upon the same base, and on the same side of it, are between the same parallels

Let the equal triangles ABC, DBC be upon the same base BC and on the same side of it *then they shall be between the same parallels*

Join AD

Then AD shall be parallel to BC

For, if possible, let AD be not parallel to BC, through the point A draw AE parallel to BC, meeting BD or BD produced at E

Join EC

Then the triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE [I. 37.

But the triangle ABC is equal to the triangle DBC [*Hyp.*

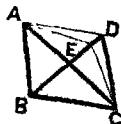
Therefore also the triangle DBC is equal to the triangle EBC, [*Ax* 1.

the greater triangle equal to the less . which is impossible

Therefore AE is not parallel to BC

In the same manner it may be demonstrated that no other straight line drawn from A but AD is parallel to BC,

therefore AD is parallel to BC

Wherefore, *equal triangles* &c. Q. E. D.

EXERCISES

1. The trapezoid, which is divided equally by one of its diagonals, is a parallelogram

2. The quadrilateral which is divided equally by each of its diagonals is a parallelogram

3. The diagonals of a quadrilateral ABCD cut each other at E. If the triangle AEB be equal to the triangle DEC, shew that AD is parallel to BC

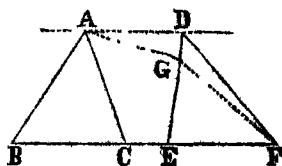
Proposition 40 Theorem.

Equal triangles upon equal bases, in the same straight line and on the same side of it, are between the same parallels

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and on the same side of it: *then they shall be between the same parallels*

Join AD

Then AD shall be parallel to BF.

For, if it is not parallel, through A draw AG parallel to BF, meeting ED, or ED produced, at G, [I. 31. and join GF

Then the triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG [I 38.

But the triangle ABC is equal to the triangle DEF. [*Hyp*

Therefore also the triangle DEF is equal to the triangle GEF, [*Ax.* 1.

the greater to the less, which is impossible

Therefore AG is not parallel to BF

In the same manner it may be shewn that no other straight line drawn from A but AD is parallel to BF

Wherefore, *equal triangles* &c Q E D

Def *The perpendicular on any side of a triangle considered as base from the opposite vertical angle is called an* **altitude** *of the triangle*

NOTE Every triangle has three altitudes

EXERCISES

1. Equal triangles between the same parallels are upon equal bases

2 Join AE, AF, and prove Prop. 40 by applying Prop 39

3. Triangles having equal bases and altitudes are equal.

4 Equal triangles upon equal bases, in the same straight line, are on opposite sides of the line; shew that the line joining the vertices is bisected by the line containing the bases

5 The quadrilateral formed by joining the middle point of a side of a triangle with the middle points of the other two sides is a parallelogram which is equal to half the triangle.

6 In Ex 2 Prop. 33, the parallelogram is half the quadrilateral

7 The straight line joining the middle points of the non-parallel sides of a trapezoid is parallel to the parallel sides and is equal to half their sum

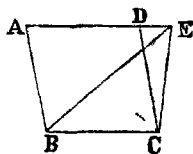
8 The straight line drawn parallel to the parallel sides of a trapezoid through the middle point of the line joining the middle points of those sides is half their sum

Proposition 41 Theorem

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE *then the parallelogram* ABCD *shall be double of the triangle* EBC

Join AC

Then the triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE [I 37

But the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects the parallelogram [I 34

Therefore the parallelogram ABCD is also double of the triangle EBC

Wherefore, *if a parallelogram* &c Q E D

EXERCISES

1 To describe a right-angled isosceles triangle equal to a given square

2 If a parallelogram and a triangle on equal bases have the same altitude, prove that the parallelogram is double of the triangle

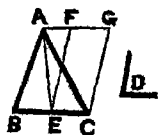
3 The two triangles whose common vertex is any point within a parallelogram, and whose bases are two of the opposite sides of the parallelogram, are together equal to half the parallelogram

4 To construct a rectangle equal to a given triangle.

Proposition 42. Problem.

To describe a parallelogram that shall be equal to a given triangle and have one of its angles equal to a given rectilineal angle

Let ABC be the given triangle, and D the given rectilineal angle . it is required to describe a parallelogram that shall be equal to the given triangle ABC. and have one of its angles equal to D

Bisect BC at E, [I 10
and join AE , at the point E, in the straight line CE, make the angle CEF equal to the angle D , [I 23
through A draw AFG parallel to EC, and through C draw CG parallel to EF [I. 31

Then FECG *is the required parallelogram* [*Def* 40

Because BE is equal to EC, [*Cons.*
the triangle ABE is equal to the triangle AEC,

because they are upon equal bases BE, EC, and between the same parallels BC, AG [I. 38.

Therefore the triangle ABC is double of the triangle AEC. But the parallelogram FECG is also double of the triangle AEC, because they are upon the same base EC, and between the same parallels EC, AG [I. 41

Therefore the parallelogram FECG is equal to the triangle ABC , [*Ax* 6.
and it has one of its angles CEF equal to the given angle D [*Cons.*

Wherefore, *a parallelogram* FECG *has been described equal to the given triangle* ABC, *and having one of its angles* CEF *equal to the given angle* D Q E F

Def A parallelogram which has a right-angle is called a ***rectangle*** or oblong

QUESTIONS FOR EXAMINATION

1 Shew that a rectangle has all its angles right-angles

2 Shew how a rectangle equal to a given triangle can be constructed.

EXERCISES

1. To describe a triangle equal to a given parallelogram, and having an angle equal to a given rectilineal angle

2 To describe a parallelogram which both in perimeter and in area shall be equal to a given triangle.

EXERCISES

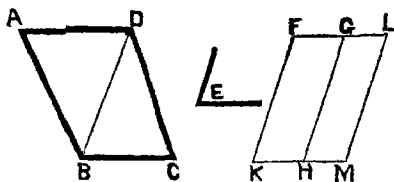
1. To construct a triangle of given altitude and which shall be equal to a given triangle

2 On a given straight line describe a triangle equal to a given parallelogram and having an angle equal to a given rectilineal angle

Proposition 45 Problem

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given rectilineal angle

Let ABCD be the given rectilineal figure, and E the given rectilineal angle, it is required to describe a parallelogram that shall be equal to ABCD, and shall have an angle equal to the given angle E.

Join DB, and describe the parallelogram FH equal to the triangle ABD, and having the angle FKH equal to the angle E and to the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E. [I 44

Then the figure FM *shall be the parallelogram required*

Because the angle E is equal to each of the angles FKH, GHM, [*Cons.*

therefore the angle FKH is equal to the angle GHM [Ax 1

Add to each of these equals the angle KHG,

therefore the angles FKH, KHG are together equal to the angles GHM, KHG [Ax 2

But FKH and KHG are together equal to two right angles [I 29

Therefore KHG, GHM are together equal to two right angles

And because at the point H in the straight line GH, the two straight lines KH, MH, upon the opposite sides of it, make the adjacent angles together equal to two right angles, therefore HM is in the same straight line with KH. [I. 14.

And because the straight line HG meets the parallels KM, FG, therefore the alternate angles MHG, HGF are equal [I. 29

Add to each of these equals the angle HGL;
therefore the angles MHG, HGL, are equal to the angles HGF, HGL. [Ax. 2.

But MHG, HGL are together equal to two right angles; [I 29
therefore HGF, HGL are together equal to two right angles

Therefore GL is in the same straight line with FG [I 14.
And because KF is parallel to HG, and HG to ML, [Cons
therefore KF is parallel to ML, [I 30
and KM, FL have been proved to be parallel,
therefore FM is a parallelogram. [Def 10
And because the triangle ABD
is equal to the parallelogram HF, [Cons.
and the triangle DBC to the parallelogram GM [Cons.
therefore the whole rectilineal figure ABCD
is equal to the whole parallelogram FM. [Ax 2

Wherefore, *the parallelogram* FM *has been described equal to the given rectilineal figure* ABCD, *and having the angle* FKM *equal to the given angle* E Q E F

COR From this it is manifest, how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle, and so on [I 44

EXERCISES

1 To describe a rectangle equal to a given rectilineal figure.

2 To describe a rhombus equal to a given parallelogram

3 On the base of an equilateral triangle construct a rectangle equal to the triangle

Proposition 46 Problem.

To describe a square upon a given straight line.

Let AB be the given straight line : it is required to describe a square upon AB.

From the point A draw AC at right angles to AB ; [I 11

and make AD equal to AB ; [I. 3

through the point D draw DE parallel to AB ; and through B draw BE parallel to AD meeting DE at E (I 31), ADEB *shall be a square*

For, ADEB is by construction a parallelogram ;

therefore AB is equal to DE, and AD to BE [I 34.

But AB is equal to AD [*Cons.*

Therefore the four sides BA, AD, DE, EB are equal, [*Ax* 1

and the quadrilateral is equilateral

Likewise all its angles are right angles

Because the angles BAD, ADE are together equal to two right angles. [I 29

and BAD is a right angle, [*Cons*

therefore also ADE *is a right angle* [*Ax* 3

Therefore each of the angles ABE, BED is a right angle [I 34

Therefore the figure ADEB is rectangular ,

and it has been shewn to be equilateral

Therefore *it is a square,* [*Def* 31

and it is described on the given straight line AB Q E F

Cor 1 Every parallelogram which has one right angle has all its angles right angles

Cor 2 Squares on equal straight lines are equal

Cor 3 Equal squares are on equal straight lines

Exercises

1 To construct a rectangle whose sides shall be equal to two given straight lines

2 If in the sides of a square, points be taken at equal distances from its four angular points in succession, the straight lines which join these points in the same order will form a square.

3 Describe a square equal to four times a given square.

Proposition 47 Theorem.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: *then the square described upon the side* **BC** *shall be equal to the squares described upon the sides* **BA, AC.**

On BC describe the square BDEC, and on AB, AC describe the squares GB, HC [I. 46.
through A draw AL parallel to BD or CE; [I 31.
and join AD, FC.

Then because the angle BAC is a right angle, [*Hyp.*
and the angle BAG is a right angle, [*Def* 31
the two straight lines AC, AG on the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles,
therefore CA is in the same straight line with AG [I. 14.
Likewise, AB and AH are in the same straight line
Now the angle DBC is equal to the angle FBA,
for each of them is a right angle. [*Ax.* 11.
To each add the angle ABC.

Then the angle DBA is equal to the angle FBC.
And because in the triangles ABD and FBC
the two sides AB, BD are respectively equal to the two sides FB, BC, [*Def* 31.
and the angle ABD is equal to the angle FBC,
therefore the triangle ABD is equal to the triangle FBC [I 4.
But the parallelogram BL is double of the triangle ABD. [I 41
Also the square GB is double of the triangle FBC. [I 41.
But the doubles of equals are equal to one another [*Ax.* 6.
Therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shewn that the parallelogram CL is equal to the square CH.
Therefore the whole square BDEC is equal to the two squares GB, HC.
And the square BDEC is described on BC, and the squares GB, HC, on BA, AC.
Therefore the square described on the side BC, is equal to the squares described on the sides BA, AC

Wherefore, *in any right-angled triangle* &c. Q. E. D.

Proposition 46 Problem.

To describe a square upon a given straight line.

Let AB be the given straight line it is required to describe a square upon AB.

From the point A draw AC at right angles to AB ; [I 11
and make AD equal to AB ; [I 3.
through the point D draw DE parallel to AB ; and through B draw BE parallel to AD meeting DE at E (I 31), ADEB *shall be a square*

For, ADEB is by construction a parallelogram ;
therefore AB is equal to DE, and AD to BE [I 34.

But AB is equal to AD [*Cons.*

Therefore the four sides BA, AD, DE, EB are equal, [*Ax* 1.
and the quadrilateral is equilateral

Likewise all its angles are right angles

Because the angles BAD, ADE are together equal to two right angles [I 29
and BAD is a right angle [*Cons*

therefore also ADE is a right angle [*Ax* 3.

Therefore each of the angles ABE, BED is a right angle [I 34.

Therefore the figure ADEB is rectangular ,
and it has been shewn to be equilateral

Therefore *it is a square,* [*Def* 31.
and it is described on the given straight line AB Q E F

Cor 1 Every parallelogram which has one right angle has all its angles right angles

Cor 2 Squares on equal straight lines are equal

Cor 3 Equal squares are on equal straight lines

EXERCISES

1 To construct a rectangle whose sides shall be equal to two given straight lines

2 If in the sides of a square, points be taken at equal distances from its four angular points, in succession, the straight lines which join these points in the same order will form a square.

3 Describe a square equal to four times a given square

Proposition 47 Theorem.

In any right-angled triangle the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: *then the square described upon the side BC shall be equal to the squares described upon the sides BA, AC*

On BC describe the square BDEC, and on AB, AC describe the squares GB, HC. [I. 46.
through A draw AL parallel to BD or CE, [I. 31.
and join AD, FC.

Then because the angle BAC is a right angle, [*Hyp.*
and the angle BAG is a right angle, [*Def.* 31.
the two straight lines AC, AG on the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles,
therefore CA is in the same straight line with AG. [I. 14.
Likewise, AB and AH are in the same straight line.
Now the angle DBC is equal to the angle FBA,
for each of them is a right angle [*Ax.* 11.
To each add the angle ABC.

Then the angle DBA is equal to the angle FBC.
And because in the triangles ABD and FBC
the two sides AB, BD are respectively equal to the two sides FB, BC, [*Def.* 31.
and the angle ABD is equal to the angle FBC,
therefore the triangle ABD is equal to the triangle FBC [I. 4.
But the parallelogram BL is double of the triangle ABD [I. 41.
Also the square GB is double of the triangle FBC. [I. 41.
But the doubles of equals are equal to one another [*Ax.* 6.
Therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shewn that the parallelogram CL is equal to the square CH.
Therefore the whole square BDEC is equal to the two squares GB, HC
And the square BDEC is described on BC, and the squares GB, HC, on BA, AC.
Therefore the square described on the side BC, is equal to the squares described on the sides BA, AC

Wherefore, *in any right-angled triangle &c.* Q. E. D.

EXERCISES

1 If one of the acute angles of a right-angled triangle be double of the other, the square on the hypotenuse is equal to four times the square on the smallest side

2 The square on the diagonal of a square is double of the given square.

3 Given a straight line, construct on it as hypotenuse a right-angled triangle so that the square on the largest side may be equal to eight times the square on the smallest side

4 If two right angled triangles have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, the triangles shall be equal in every respect

5 If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the difference of the squares described on the two parts of the hypotenuse is equal to the square on the other side

6. The points F, A, K are collinear

7 Describe a square equal to two given squares

8 Describe a square equal to the difference of two given squares

9 Divide a straight line into two parts, such that the sum of their squares shall be equal to a given square

10 Divide a straight line into two parts so that the square on one part may be double of the square on the other part

11 The sum of the squares on the four sides of a rhombus are together equal to the squares on the two diagonals

12 Join GH, FD and KE, and prove that the three triangles thus formed are each equal to the triangle ABC, and to one another

13 If a straight line be at right angles to a finite straight line, the difference of the squares on the straight lines which join a point in the former with the extremities of the latter is the same, whatever point be taken

14 In a right-angled triangle, the sum of the squares on the three sides of the triangle is equal to eight times the square on the line drawn from the right angle to the point of bisection of the hypotenuse

15 If one angle of a triangle be a right angle, and another equal to two thirds of a right angle, prove that the equilateral triangle described on the hypotenuse, is equal to the sum of the equilateral triangles described on the sides containing the right angle

16 If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

Proposition 48. Theorem.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides AB, AC · then *the angle* BAC *shall be a right angle.*

From the point A draw AD at right angles to AC ; [I. 11
and make AD equal to BA ; [I 3
and join DC.

Then because DA is equal to BA, the square on DA is equal to the square on BA To each of these equals add the square on AC

Therefore the squares on DA, AC are equal to the squares on BA, AC. [*Ax.* 2.

But because the angle DAC is a right angle, [*Cons*
therefore the square on DC is equal to the squares on DA, AC. [I. 47.

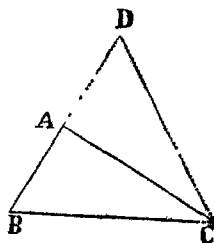
And, by the hypothesis, the square on BC is equal to the squares on BA, AC

Therefore the square on DC is equal to the square on BC [*Ax* 1.

therefore also the side DC is equal to the side BC.

And because the side DA is equal to the side BA, [*Cons*
and the side AC is common to the two triangles DAC, BAC :
the two sides DA, AC are equal to the two sides BA, AC each to each , and the base DC has been proved to be equal to the base BC ,

therefore the angle DAC is equal to the angle BAC. [I. 8.
But DAC is a right angle ; [*Cons.*
therefore BAC is also a right angle. [*Ax* 1.

Wherefore, *if the square* &c Q. E. D.

Exercises

1. If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle contained by these sides is acute.

2. If the square on one side of a triangle be greater than the sum of the squares on the sides containing the opposite angle, this angle is obtuse.

3. If in the triangle ABC, BC be double of AB and the square on AC be three times the square on AB, the angle BAC is a right angle.

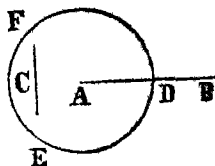
Granting that a circle may be described from any centre, with a radius equal to any finite straight line (See Notes), Prop. 3 may be proved in the following manner

Proposition 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater

It is required to cut off from AB, the greater, a part equal to C, the less

With A as centre and C as radius describe the circle DEF cutting AB at D [*Post.* 3.

Then AD (being a radius of the circle) is equal to C

Wherefore, *from* AB, *the greater of two given straight lines, a part* AD *has been cut off equal to* C *the less.*

Which was to be done

Proposition 5 Theorem

The angles at the base of an isosceles triangle are equal to one another *

Let ABC be an isosceles triangle, having the side AB equal to the side AC.

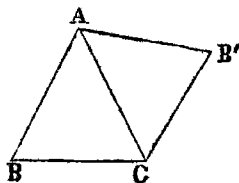
Then the angle ACB *shall be equal to the angle* ABC.

Imagine the triangle ABC to be turned about AC until it is placed again in its own plane, so that AB takes the position AB′

Because AB is equal to AC, [*Hyp*

and AB is equal to AB′,

therefore AC is equal to AB′

Therefore in the two triangles ABC, ACB,′

the sides AB, AC are respectively equal to the sides AC, AB′,
and the angle BAC is equal to the angle CAB′;

Therefore the angle ABC is equal to the angle ACB′. [I. 4.

But the angle ACB′ is equal to the angle ACB.

Therefore the angle ABC is equal to the angle ACB.

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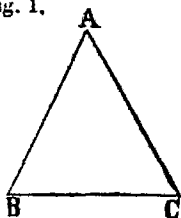
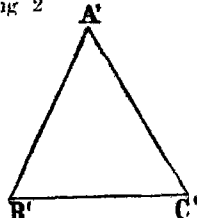
* The second part of this Proposition is omitted and is given as Cor 2 to Proposition 13

ANOTHER PROOF.

Fig. 1.

Fig 2

A'

B' C'

Imagine the triangle ABC to be moved in space and in the same plane until it comes to the position of Fig 2; and designate the angular points A', B', C', so that A' is equal to A, B to B and C' to C.

Then because in the triangles ABC, A'B'C',

AB is equal to A'C', for A'C' is equal to AC,
AC is equal to A B', for A'B' is equal to A B
and the angle BAC is equal to the angle B'A'C',

therefore the angle ACB is equal to the angle A'B'C' [I 4.

But the angle A'B'C' is equal to the angle ABC

Wherefore, the angle ACB is equal to the angle ABC Q E D

Proposition 6 Theorem

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another

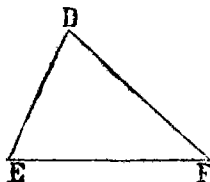
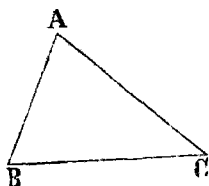
Let ABC be a triangle having the angle ACB equal to the angle ABC.

Take up the triangle, turn it over and replace it so that the point C may fall where B was, and the line CB along the straight line BC, then B would fall where C was

Then because the angle ABC is equal to the angle ACB, the line CA shall be along BA, and BA along CA; the point A shall fall on its former position and the lines CA, BA shall coincide with the lines BA, CA

Therefore AB is equal to AC Q. E D.

Proposition 8. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the triangles will be identically equal

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and also the base BC equal to the base EF

Then the triangle ABC *shall be identically equal to the triangle* DEF

Suppose the triangle DEF be applied to the triangle ABC, so that EF coincides with BC and the vertex D falls on the side of BC opposite to the side on which A is, join AD

CASE 1 When AD passes across BC

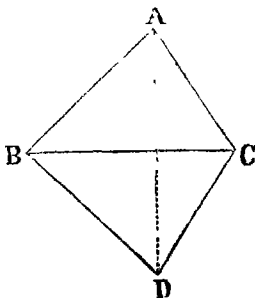
Then in the triangle ABD, because BD is equal to BA therefore the angle BAD is equal to the angle BDA [I 5

And in the triangle ACD, because DC is equal to AC, therefore the angle CAD is equal to the angle CDA [I 5

Therefore the sum of the angles BAD and CAD is equal to the sum of the angles BDA and CDA [*Ax* 2

∴ the angle BAC is equal to the angle BDC, which again is equal to the angle EDF

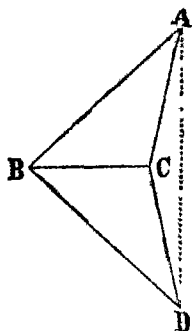
∴ the angle BAC is equal to the angle EDF

∴ by Prop 4, the triangles ABC and DEF are identically equal

CASE 2. When AD passes outside BC

As in the preceding case, the angle BAD is equal to the angle BDA, and the angle CAD is equal to the angle CDA

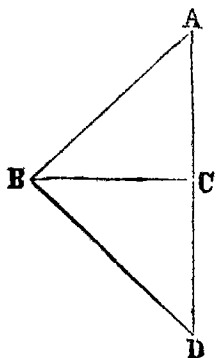
The difference of these equal angles are equal [*Ax* 3.

Therefore the angle BAC is equal to the angle BDC, which again is equal to the angle EDF,
therefore the angle BAC is equal to the angle EDF.

∴ by Prop 4, the triangles ABC, DEF are identically equal

CASE 3 When AD coincides with AC, CD.

In the triangle ABD, the angle BAD is equal to the angle BDA, therefore the angle BAC is equal to the angle BDC, which again is equal to EDF

∴ the angle BAC is equal to the angle EDF

∴ by Prop 4, the triangles ABC and DEF are identically equal

NOTES ON BOOK I.

DEF 1. Point (Lat *punctum, pungere*, to prick) means the sharp end of any thing or a mark made by it This term cannot convey the exact notion of what is to be understood by a point in Geometry. A point in Geometry expresses the idea of the position of a very minute particle when its magnitude is not taken into consideration

DEF 2 LINE (Lat *linea*, a linen thread), uniting the positive idea of length with the negative one of defect of breadth and thickness, expresses the proper notion of a geometrical line If a point move in a plane, it will describe a path, this path is a line, hence lines may be straight or curved

DEF 3 This definition explains the first.

DEF 4 The following is another definition --*The shortest distance from one point to another is a straight line*

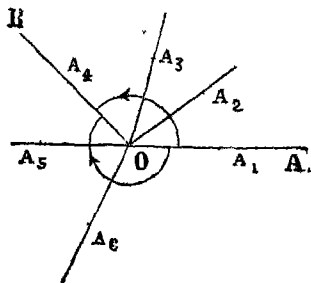
If a point move without changing its direction, it will describe a straight line The direction in which the point moves is called its *sense* If a weight be suspended by a string, the string will be stretched and will become straight It we mentally abstract from this string its thickness and breadth we obtain the notion of a straight line

DEF 5 The latter part of this definition is explanatory of DEF 2

DEF 6 A plane surface is that which is everywhere perfectly flat and even, like the surface of still water

DEF 10 The greater or less length of the arms makes no alteration in the angle

The nature of the magnitude of an angle may be illustrated in the following manner —

Let O be the extremity of a straight line OA, extending indefinitely in the direction OA Through the point O, let another straight line OA_1 be conceived to be drawn, and suppose this straight line to revolve in the same plane round its extremity O, it being supposed at the beginning of its motion to coincide with OA As it revolves from OA to OA_2, OA_3, OA_4, &c, its divergence from OA_1 or, what is the same, the angle it makes with OA, continually increases.

The magnitude of the angle AOB is the amount of the turning of OA_1 around O (that is, the amount of turning of the revolving line from its initial position) in a direction opposite to the hands of a watch as is shewn by the arrow-head, but the magnitude of the

angle may also be the amount of turning of OA_1, in the opposite *sense* *i e* in the same direction as the hands of a watch The two angles thus formed are said to be **conjugate** The smaller of the two is called the **minor conjugate**, and the greater the **major conjugate** angle The student will learn hereafter in Trigonometry, that if the convention be adopted that angles measured in one direction be considered **positive** (for instance against the hands of a watch), then angles measured in the opposite direction (that is, with the hands of a watch) shall be considered **negative**

The major conjugate angles are sometimes called the **reflex** or **re-entrant** angles

When nothing else is mentioned, the minor conjugate angle is to be understood

Def *When the arms of an angle are in the same straight line, the conjugate angles are equal, and each is then called a* ***straight angle***

Def 11 When one straight line is perpendicular to another straight line, the latter is also perpendicular to the former

Defs 24 to 30 Triangles are divided into three classes, by considering the relations of their sides, and into three other classes with reference to their angles

Def 31 In some Books on Geometry this definition is given differently "A square is a four sided plane figure having all its sides equal and one angle a right angle Because it is proved in Prop 46 Book I that if a parallelogram have one angle a right angle, all its angles are right angles

Def 33 This definition and the terms *rhomboid* are superseded by the term *parallelogram* which is a quadrilateral whose opposite sides are parallel It will be proved hereafter that if the opposite sides of a quadrilateral be equal, it must be a parallelogram

Postulates The object of the postulates is to declare that the only instruments the use of which is permitted in Geometry are the *straight edge of an ungraduated ruler and a pair of compasses*

The straight edge of a ruler is used to draw and produce straight lines

3rd Post This postulate of Euclid permits the use of a pair of compasses

Use of a pair of compasses A pair of compasses is an instrument composed of two legs united at one extremity by a joint, and the other extremities of which are pointed, the instrument is so constructed that the legs can be opened or closed so that the points of the legs may have a given distance between them

To describe a circle by Euclid's postulate fix the extremity of one of the legs at the point which is to be the centre and the extremity of the other at the point which is at a given distance from the centre, now let the latter point move around the former point, the points maintaining the same distance apart, the moving point will describe the circumference of a circle

This postulate also virtually admits that a pair of compasses may be used for the transference of distances

For when the pair of compasses is used for describing a circle, the legs are so opened that the points of the compasses are placed on *two* points, *one* at the centre and the other at a certain distance from that centre, that is, the points are placed on the extremities of a finite straight line. Now if we take away the pair of compasses without changing the angle between the legs, that is, without changing the distance between the points of the legs, and place one end of a leg on a given point and join this point to the point where the end of the other leg falls, we shall transfer a distance.

In describing a circle by Euclid's postulate we assume that one of the points of the compasses moves while the other remains fixed. In the process given above we assumed that both the ends move simultaneously.

To assume that one point of a pair of compasses can move and that both the points cannot move is an arbitrary and unmeaning restriction.

Hence the restriction may be taken away, and the following postulate may be substituted: "That a circle may be described from any centre, with a radius equal to any finite straight line." Therefore this postulate virtually admits that a pair of compasses may be used for the transference of distances.

The first Book treats of the properties of triangles and parallelograms.

A proposition when complete may be divided into six parts:— General Enunciation, Particular Enunciation, Construction, Determination, Demonstration, and Conclusion.

We divide Prop. I, Book I accordingly:—

General Enunciation	To describe an equilateral triangle on a given finite straight line.
Particular Enunciation	Let AB be the given straight line, it is required to describe an equilateral triangle on AB.
Construction	From the centre B, at the distance BA, describe the circle BCD, and from the centre A, &c.
Determination	—Then ABC shall be an equilateral triangle.
Demonstration	Because the point A is the centre of the circle BCD, therefore AC is equal to AB, and because the point B, &c.
Conclusion	Wherefore the triangle ABC is equilateral, and it is described on the given straight line AB.

Prop. 1. If the points A, B, be joined with the other point of intersection, the triangle thus formed is also equilateral.

Prop. 2. This problem as given by Euclid admits of eight different lines being drawn from the given point in different directions, because—(1) Two lines can be drawn from the given point to the two extremities of the given line. (2) The equilateral triangle may be described on either side of this line. (3) The side BD of the equilateral triangle ABD may be produced either way.

But if the given point be in the line or in the line produced, only four different lines can be drawn For the lines joining the extremities of the given line with the given point coincide with each other or are in the same straight line Therefore, in each case, the two different lines become the same

Prop 3 The part to be cut off may be cut off from any end of the greater straight line

Prop 4 The principle of superposition is employed in the proof of this proposition This is the first case of identically equal or congruent triangles, two other cases are demonstrated in Prop 8. and Prop 26

Prop 6 is the converse of Prop 5 Thus in I 5 the hypothesis is the equality of the sides, and the conclusion is the equality of the angles at the base, in I 6, the hypothesis is the equality of the angles at the base, and the conclusion is the equality of the sides When there are several hypotheses or several conclusions to a proposition, we may form more than one converse propositions Thus —If the angles formed by the base of a triangle and the sides produced be equal, the sides of the triangle are equal This proposition is true —For let BC be the base of the triangle ABC, and let AB and AC be produced to D and E respectively The angle CBD is equal to the angle BCE The angles ABC and CBD are together equal to two right angles and also the angles ACB and ECB are together equal to two right angles (I 13) Therefore the angles ABC and CBD are together equal to the angles BCA and BCE (Ax 1) Therefore the angle ABC is equal to the angle ACB (Ax 3) Therefore AB is equal to AC (I 6)

In Prop 6, the method of demonstration is called *indirect* The proposition is proved to be true, by shewing that any supposition to the contrary would lead to an absurdity

This kind of proof is considered inferior to *direct* demonstration, because it only proves that a thing *must* be so, but fails to show *why* it must be so, whereas *direct* proof not only shows that the thing *is* so, but also *why* it is so 14th, 19th, 25th, and 40th propositions are proved indirectly

Prop 7 is required only in the demonstration of Prop 8, we have given a direct demonstration of the 8th, (see page 71) and Prop 7 may be dispensed with altogether

Prop 8 is the second case of congruent triangles Euclid only shews that the angles opposite to the bases are equal, but he always proves the equality of the areas of two coincident triangles by using I 4

Converse propositions are not universally true The converse of Prop 8—"If the three angles of one triangle be respectively equal to the three angles of another triangle, the three sides of the former shall be equal to the three sides of the latter," is not universally true

Prop 9 The equilateral triangle DEF is described on the side DE *remote* from A, for if it be described on the same side in which A

is, then the vertex F may coincide with A and the angle cannot be bisected

If BA and AC be in the same straight line, this problem then becomes the same as Prop 11, which will then become the same as to draw a straight line which shall bisect an angle equal to two right angles

By means of this problem, an angle may be divided into any number of equal parts denoted by the successive powers of the number 2, that is, in 2, 4, 8, 16, &c equal parts

Prop 10 By this problem a straight line may be divided into four, eight, sixteen, thirty two, &c equal parts In this and in the following proposition an isosceles triangle instead of an equilateral triangle would answer the purposes of the solution In the demonstration nothing is inferred from the equality of the base with the sides

Prop 11 If the point be at the extremity of the given straight line, the straight line is to be produced

Prop 12 The student should bear in mind the distinction between the expression *at right angles* and *perpendicular* A straight line is drawn *at right angles* to another straight line when it is drawn from a point in the latter (as in I 11) A straight line is *perpendicular* when it is drawn from a point *without* another The *distance* between a point and a straight line is the *shortest line* which can be drawn from the point to the line; and we can prove that the perpendicular from the point to the line is the *shortest straight line*

Prop 13. Each of the angles DBA and CBA is the supplement of the other

COR 2 The sum of each base angle and its supplementary angle (the adjacent angle on the other side of the base) is equal to two right angles (Cor 1) From these equals take away the base angles, (which are equal by I 5) Therefore the angles on the other side of the base are equal. [*Ax.* 3.

COR 3 If it be possible, let the segment AB be common to the two straight lines ABC, ABD

From the point B, draw any straight line BE

Then, because ABC is a straight line, [*Hyp*

the angles CBE, EBA, are together equal to two right angles [I 13.

Also, because ABD is a straight line, [*Hyp*

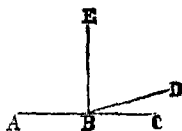
the angles DBE, EBA are together equal to two right angles

Therefore the angles ABE and DBE are together equal to the angles ABE and EBC

Take away the common angle ABE

Therefore the angle DBE is equal to the angle CBE, [*Ax* 3

the less to the greater, which is impossible [*Ax* 9

Wherefore *two straight lines cannot have a common segment.*

Prop 14 is the converse of Prop 13.

Prop 15. The converse of this proposition is not proved by Euclid, viz —If four straight lines meet at a point and make the opposite vertical angles respectively equal to each other, each pair of opposite lines shall be in the same straight line [Ex 1.

Prop 16 A new axiom is used in the demonstration of this and some other subsequent propositions, *viz* —If two things be equal to one another, and one of them be greater than a third, the other is also greater than the third

Prop 17. This proposition and the 16th are included in the 32nd

Props 18, 19 In each of these propositions the *hypothesis* is stated before the conclusion Prop 19 is the converse of Prop 18 Prop 19 bears the same relation to Prop 18 as Prop 6 to Prop 5

Prop 20 We may deduce the following corollary from this proposition —A straight line is the shortest distance between two points For the straight line BC is always less than the straight lines BA and AC, whatever be the position of A

Prop 23 The eleventh proposition is a particular case of this proposition

Prop 24 In the construction of this proposition the Greek editors have omitted the word *which is not greater*, but these are absolutely necessary to prevent a diversity of cases, the point H might fall above the line BC, or below it, as well as upon this line

Prop 24 bears the same relation to Prop 25 as Prop 4 to Prop 8

Prop 26 forms the third case of congruent triangles

A triangle has *six* parts, -three sides and three angles When two triangles agree in three of the above-mentioned six parts, the other three may be obtained (excepting the case when the three angles are given), combining any three of six parts, we get the following six cases —

(i) Two sides and the angle between them
(ii) Two angles and the side between them
(iii) Two sides and the angle opposed to one of them
(iv) Two angles and the side opposed to one of them
(v) The three sides
(vi) The three angles

(i) is proved in Prop 4

(ii) and (iv) in Prop 26, (v) in Prop 8 In (vi) the triangles are not necessarily equal

(iii) is proved in Prop I, Appendix.

From proposition 26 and the principles established before, it easily follows, that a line being drawn from the vertex of a triangle to the base, if any two of the following equalities be given (except the first two) the others may be inferred —

(1) The equality of the sides of the triangle

(2) The equality of the angles at the base

(3) The equality of the angles under the line drawn, and the base.

(4) The equality of the angles under the line drawn, and the sides

(5) The equality of the segments of the base

NOTE The line drawn is the axis of symmetry

Some of the cases have already been given (Ex 4, Prop 9), (Ex 4, Prop 10), (Ex 2, Prop 12), (Ex 4, Prop 26) &c

The student may try with advantage the other cases.

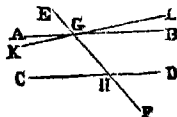
DEF 37 It is possible for two straight lines (when not in the same plane) never to meet when produced, and yet not be parallel Two things are indispensably necessary to establish the parallelism of two straight lines, (1) that they be in the same plane, and (2), that when indefinitely produced they never meet As in the first six books of the Elements all the lines which are considered are supposed to be in the same plane, it will be only necessary to attend to the latter criterion

Prop 27 The crooked lines EBG and FDG are considered straight lines and the figure EBGDF, a triangle, for the sake of the argument

Prop 29 is the converse of Prop 27 and Prop 28

Prop 29 The 12th axiom of Euclid on which this proposition depends seems to be improperly placed among the axioms, as it is not self evident Dr Simson has proved this theorem by assuming two definitions and one axiom and demonstrating five subsidiary propositions

The 12th axiom may, however, be admitted, as a corollary to Prop 28

Professor Playfair has avoided this axiom and has proved Prop 29 by assuming the following simple axiom – "Two straight lines which intersect one another, cannot be both parallel to the same straight line" If the angle AGH be not equal to the angle GHD, one of them must be greater Let AGH be greater than GHD Make the angle KGH equal to the angle GHD, and produce KG to L KL will be parallel to CD Therefore two straight lines are drawn from the same point G, parallel to CD, and yet not coinciding with one another, which is impossible. (Playfair's axiom).

Prop 30 We may easily prove the case of this proposition when EF is not between AB and CD, but on either side of both

Prop 31 In the construction of this proposition the words *on the opposite side of* AD must be added, for otherwise the angle EAD may be made on the same side of AD, and then the problem would fail

Prop 32 In addition to the seven corollaries given after this proposition we may deduce the following —

8 If one angle of a triangle be a right angle, the sum of the other two is a right angle

9 If two angles of a triangle be given, the third is given.

10 If one angle of a triangle be greater than the sum of the other two, it is obtuse, and if less, acute.

Prop 32, Cor 2 The exterior angles are to be taken at each of the angular points Either of the adjacent sides at each angular point is produced to form the exterior angles, for the two angles thus obtained are equal, by I 15

Prop 33 The words *towards the same parts* are necessary. The enunciation of this proposition may be more clearly expressed thus — The straight lines which *without crossing each other*, join the extremities of two equal and parallel straight lines, are themselves equal and parallel '

Prop. 34 If the other diagonal be drawn, it may be proved that the diagonals of a parallelogram bisect each other, as well as bisect the area of the parallelogram

Prop 35 By the word *equal*, the parallelograms are to be considered equal in *area*.

Prop 38 If the point E coincide with C, and D with A, then each of the two angles ACB and DEF will be supplemental to the other Hence we may deduce the following —

If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, the two triangles are equal in area.

Props 39, 40 If the vertices of all the equal triangles described on the same base (Prop 39), or on equal bases (Prop 40), be joined, the line thus formed will be a straight line, and is called the locus (page 96) of the vertices of equal triangles on the same base, or on equal bases.

We may prove I 40, directly —Join BD and CD The triangle DBC is equal to the triangle DEF, (I 38) but the triangle ABC is equal to the triangle DEF, therefore the triangle ABC is equal to the triangle DBC (Ax 1), therefore AD is parallel to BC. (I 39)

Prop 41 *The area of a triangle is half of the area of the right-angled parallelogram having the same base and altitude* —for, the area of any parallelogram is equal to that of a right-angled parallelogram having the same base and altitude.

Prop. 47. This proposition was discovered by Pythagoras. In the construction of this proposition, Euclid has considered the case in which the three squares are described on the *outer* side of the triangle ABC There are five other cases .

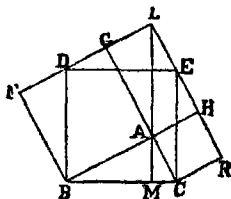
1. The three squares on the three *interior sides* of the triangle.

2 The two smaller squares on the *exterior sides*, and the greater on the *interior side*

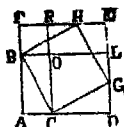
3 The two smaller squares on the *interior sides*, and the greater on the *exterior side*

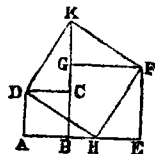
4 The greater and one of the smaller squares on the *exterior sides*, and the other on the *interior side*

5 The greater and one of the smaller squares on the *interior sides*, and the other on the *exterior side*

We shall prove the second case Let FA and AR be the two squares on BA and AC respectively Draw BD at right angles to BC meeting FG or FG produced at D, and draw CE at right angles to BC meeting RH produced or RH at E Because in the triangles BFD and BAC, the angles FBD and DFB are respectively equal to the angles ABC and CAB, also the base FB is equal to the base BA therefore DB is equal to BC (I 26) Likewise EC is equal to BC, ∴ DB is equal to EC, but DB is parallel to EC, because the angles DBC and ECB are together equal to two right angles (I 28), therefore DE is equal and parallel to BC (I 33) Therefore DBCE is a square and it is described on BC Through A draw LM parallel to BD or CE, meeting FG in L and BC in M Join D, A. BG is double of the triangle DAB, also DM is double of the same triangle Therefore BG is equal to DM Likewise we can prove CH to be equal to EM Therefore &c &c

Likewise, the student may advantageously adapt the demonstration given in the text to the other four cases

This proposition may be demonstrated in many other different ways We shall here subjoin two very interesting demonstrations —

1. Produce AC to D making CD equal to AB On AD describe the square ADEF From DE and FE cut off DG and EH each equal to AC Through C draw CR parallel to AF or DE, and through B draw BOL parallel to FE or AD Join CG, GH, and HB.

Because AC, DG, EH, FB are equal, and BA, CD, GE, HF are equal and the angles at A, D, E and F are right angles, therefore the triangles ABC, DCG, EGH, and FHB are equal (I 4) and BC, CG, GH and HB are equal Also the angle CBA is equal to the angle GCD To these equals add the angle BCA Therefore the angles BCA and GCD are together equal to the angles ABC and BCA, which are together equal to one right angle (I 32) Therefore the angle BCG is a right angle (I 13) Therefore BCGH is a square and it is described on BC Because CD, AB, DL are equal, therefore CL is a square, therefore it is the square on CD and equal to the square on AB Likewise we can prove that BR is equal to the square on AC AO and OE are together double of AO and therefore quadruple of ABC Also the triangles BAC, CDG, GEH, HFB are together quadruple of ABC, therefore these are equal to AO and OE Take these equals from FD, then HC is equal to FO and OD Therefore the square on BC is equal to the squares on AB and AC

2 Let ABCD and BEFG be any two squares, let them be so placed that their bases AB and BE may be in the same straight line Make AH and CK each equal to BE Join DK, KF, FH and HD Because AH is equal to BE, therefore AB is equal to HE (Ax 3), CK, AH, BE, BG are equal therefore GK, CB, AB, DC are equal therefore GK, DC, DA, HE are equal GF, CK, AH, EF and the angles KGF, DCK, DAH, HEF are equal Therefore the triangles KGF, DCK, DAH, HEF are equal (I 4), and FK, KD, DH, HF are equal, by (I 32) DHFK is a square

Because DCK and KGF are together equal to DAH and HEF take these equals from the whole figure AEFKD, then DB and GE are together equal to DF But DB is the square on AD, GE is equal to the square on AH and DF is the square on DH Therefore, &c , &c.

Prop 47 is a particular case of the proposition of Pappus, which we have given as Ex 2, Prop. 35

Prop. 48 is the converse of Prop 47

If the sides of a triangle be represented by three straight lines whose lengths are 3, 4, 5 units; the angle, which is contained by the straight lines which are represented by 3 and 4, is a right angle.

For $4^2+3^2=16+9=25=5^2$.

The ancients have given several rules for finding three whole numbers which shall represent the sides of a right angled triangle.

The following are some of the rules given:—

Rule of Pythagoras.

1. Take any odd number.

2. Subtract unity from the square of this number, and take half the difference.

3. Add unity to the square of the number, and take half the sum.

The numbers so found, with the original number, represent the lengths of the sides of a right angled triangle.

(1) Let the odd number be 7.

(2) $\frac{1}{2}(7^2-1)=24$ (3) $\frac{1}{2}(7^2+1)=25$

Then 7, 24 and 25 are the three numbers.

For $24^2+7^2=576+49=625=25^2$.

Rule of Plato.

1. Take any even number.

2. Take the square of half the number, and subtract unity.

3. Take the square of half the number and add unity.

The numbers so found, with the original number, represent the lengths of the sides of a right-angled triangle.

(1) Let the even number be 6.

(2) $3^2-1=8$ (3) $3^2+1=10$

Then 6, 8, 10 are the three numbers.

For $6^2+8^2=36+64=100=10^2$.

Rule of Euclid.

1. Let two numbers, either both odd or both even, be assumed such that their product shall be a perfect square.

2. Take half the difference of these numbers.

3. Add this half difference to the less.

The numbers thus found, together with the number whose square is the product of the assumed numbers, represent the lengths of the sides of a right-angled triangle

1 (1) Let the assumed numbers be 16 and 4, whose product, 64, is the square of 8.

(2) $\frac{1}{2}(16-4)=6$ (3) $6+4=10$.

Therefore 6, 10, 8 are the numbers required

2 (1) Let the assumed numbers be 3 and 27, whose product, 81, is the square of 9

(2) $\frac{1}{2}(27-3)=12$ (3) $12+3=15$

Therefore 15, 12, 9 are the required numbers

Another Rule.

Take any two numbers.

1. Take the sum of their squares

2 Take the difference of their squares

3 Take twice their product

The three numbers so found represent the sides of a right-angled triangle

Example

Let the numbers be 3 and 2.

(1) $3^2+2^2=13$.   (2) $3^2-2^2=5$ (3) $2\times2\times3=12$

Therefore 12 5, 13 are the required numbers

The First Book of Euclid's Elements may be divided into three parts The first part (Props 1 to 26) treats of the origin and properties of triangles, with regard to the equality and inequality of the sides and angles The second part (Props 27 to 34) treats of the properties of parallel lines and parallelograms The third part (Props 35 to 48) treats about the relation of triangles and parallelograms in regard to area, and the equality of the squares on the sides of a right-angled triangle to the square on the hypotenuse

QUESTIONS FOR EXAMINATION ON BOOK I

1. What is the difference between *Plane Geometry* and *Geometry of Planes?*

2. State what a point possesses positively.

3 When is one straight line said to *cut*, and when to *meet* another?

4. If the angle between two straight lines be continually diminished, to what state do they approach?

5 Figures (*i e*, surfaces) are different according to their containing lines, name a figure contained by one line only, and accurately define it

6 Name the three kinds of angles, explain their names and accurately define them

7 When is a straight line said to be drawn *at right angles* and when *perpendicular*, to a given straight line?

8 Name a figure contained by two lines

9 Show that triangles are of three kinds, according to the equality of their sides, and name the three kinds.

10 Name the three kinds of triangles according to their angles How many acute angles *must* every triangle have?

11 What name is given to a four sided figure, the sides of which are neither parallel nor equal? Define accurately "square," "rhombus," "oblong," "rhomboid," and state what quality they possess in common

12 What are the points of resemblance and difference between *a square* and *a rhombus*?

13 Can the diagonals of any quadrilateral figure be called diameters? why can they be in parallelograms?

14 Two straight lines may be produced ever so far both ways without meeting and yet not be parallel Mention one familiar instance of this

15. Define *distance* of a point from a straight line

16 Enumerate the principles of construction assumed by Euclid

17 State the axiom which immediately results from the definition of a right angle

18. Explain "data," "corollary," "enunciation," "hypothesis," and "conclusion or predicate"

19 Distinguish accurately between a "problem" and a "theorem," and give an instance of each

20 Of what two parts does the enunciation of a Problem and of a Theorem consist? Distinguish them in Euc I 1, 3, 4, 5, 10, 20.

21 When is one proposition said to be the converse of another?

22 Distinguish between a *direct* and *indirect* proof

23. If in Euclid (I 1) straight lines be drawn from the extremities of the given straight line to the other point of intersection of the circles so as to form another triangle, what figure will the two triangles form?

24 How many parts or elements are there in every triangle, and what are they?

25 In what cases is the following proposition true, and in what case is it false? "If two plane triangles have three similar elements in the one respectively equal to three similar elements in the other, the triangles are equal in every respect"

26 What inference would you draw as to the equality of two triangles which are equiangular to each other?

27 Why does not Euclid prove the third case of Prop 7. Book I?

28 In the construction of Euclid I 9, can we, in all cases, describe the equilateral triangle on any side of the joining line?

29 Into what equal parts can we divide an angle by Euclid I 9?

30 In what cases the lines which bisect the interior angles of plane triangles will also bisect one or more than one of the corresponding opposite sides of the triangles?

31 Into what equal parts can we divide a straight line by Euclid I 10?

32 Show by I 16 that more than one perpendicular cannot be drawn from the same point outside it to the same straight line

33 "A straight line is the shortest distance between any two points" From what proposition may this be inferred?

34. Shew that the base of a triangle being a constant (*i e*, given or measured in length), as the sides increase in length the vertical angle decreases, and *vice versâ*

35 To construct a triangle of which the sides shall be equal to three given straight lines. Is any limitation necessary? If so, why?

36 "Construct a triangle with three straight lines whose lengths are 1, 3, 4 units" Is the proposition possible?

37 Construct a triangle whose angles shall be as the numbers 1, 3, 4

38 In any triangle, shew that the sum of all the sides is greater than double of any one of them, and that the sum of all the sides is less than double the sum of any two of them

39. "Two sides and an angle in one triangle are respectively equal to two sides and an angle in the other" Are the two triangles equal in all cases? State the case, if any, when they are not equal. See Add Prop I

40 "A side and two angles in one triangle are respectively equal to a side and two angles in the other" State the case, if any, when these triangles are not equal

41 Name all the properties of parallel lines, and lines meeting them How may it be known that lines are parallel?

42 Define "adjacent angles," "opposite angles," "vertical angles," and "alternate angles", and give examples from Euclid, Book I

43 Shew that the distance between two parallel straight lines is constant

44 Define *exterior* and *interior* angles, and give examples

45 Shew that in any isosceles triangle, each angle at the base is equal to half the external vertical angle

46 Turn down the corners of a triangular piece of paper, so as to exhibit to the eye that the three angles of a triangle are equal to two right angles

47 Shew that each angle of an equilateral triangle is a third of two right angles, or two-thirds of one

48 Shew that each angle of a regular hexagon is equal to four-thirds of one right angle, and that each angle of a regular decagon is equal to eight-fifths of one right angle

49 If from any two points of a straight line, perpendiculars be let fall upon another straight line, and if these perpendiculars be equal, shew that the straight lines are parallel, but if they be unequal, the straight lines shall meet on the side of the smaller perpendicular.

50 Shew that a quadrilateral figure is a parallelogram when its opposite sides are equal, when its opposite angles are equal or when its diagonals bisect each other

51 Shew that the diagonals of a rhombus bisect its angles

52 If two triangles have two sides of the one equal to two sides of the other, each to each, and the contained angles together equal to two right angles, prove that the two triangles shall be equal

53 If either parallelogram (I 43) about the diagonal of a parallelogram be a square, the other parallelogram about the diagonal, and also the parallelogram itself are squares, if the complements be squares, determine their relation to the whole parallelogram

54 Shew that squares constructed on equal straight lines are equal

55 Show that the square on a diagonal of any square is double of it

56 Enumerate all the properties of *triangles* and *parallelograms* proved in the First Book of Euclid

57 Find (by the rule of Pythagoras) two whole numbers which together with 9 shall represent the side of a right angled triangle

58 Apply the rule of Plato to find two whole numbers which together with 8 represent the sides of a right angled triangle

59 Can we express as whole numbers the diagonals of a square whose sides are whole numbers?

60 Shew that the square on one side of a right-angled triangle is equal to the difference of the squares on the hypotenuse and on the other side

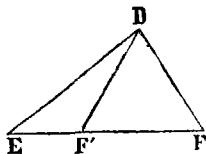
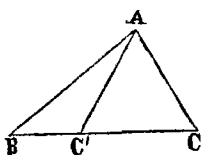
61 Find a straight line the square on which is equal to the sum of the squares on two given straight lines

62 Find a straight line the square on which is equal to the difference of the squares on two given straight lines

ADDITIONAL PROPOSITIONS, BOOK I.

Proposition I Theorem

(5TH CASE OF CONGRUENT TRIANGLES) SEE (III) PAGE 78

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides in each equal the angles opposite to the other pair of equal sides are equal or supplementary, and in the first case the triangles are congruent

Let ABC and DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE and AC to DF, and the angle ABC equal to the angle DEF

Then the angles ACB, DFE *opposite to the equal sides* AB, DE, *are equal or supplementary, and in the first case the triangles* ABC, DEF *are congruent*

For, if the triangle DEF be applied to the triangle ABC so that the point E may be on the point B, and the straight line ED on the straight line BA,

then the point D shall coincide with the point A,
because AB is equal to DE

ED coinciding with BA, EF must fall on BC,
because the angle DEF is equal to the angle ABC

Now, if the angle EDF be equal to the angle BAC,
DF shall coincide with AC, and the angle DFE shall be equal to the angle ACB and the triangles ABC, DEF shall be congruent

But if the angle EDF be not equal to the angle BAC,
let DF′, be the position of DF,
then make the angle BAC′ equal to the angle EDF′

In the two triangles ABC′ and DEF′,
$AB = DE$, $\angle ABC' = \angle DEF'$ and the $\angle BAC' = \angle EDF'$,
$\therefore AC' = DF'$, and the $\angle AC'B = \angle DF'E$ [I 26

But $F'D = FD = AC$, $\therefore AC' = AC$
$\therefore \angle AC'C = \angle ACC'$

$\therefore \angle AC'B$ or $\angle DF'E$ is supplementary to $\angle AC'C$ or $\angle ACC'$,
$\therefore \angle DF'E$ is supplementary to $\angle ACB$

Wherefore, *if two triangles have* &c Q E D.

Proposition II Theorem

The straight line joining the points of bisection of any two sides of a triangle is parallel to the remaining side and also half of it

Let ABC be a triangle, D, E the middle points of AB, AC respectively, DE *is parallel to* BC *and half of it*

Draw CF parallel to AB meeting DE produced at F

Because in the triangles ADE, CFE,

the side AE is equal to the side EC,

the angle DAE is equal to the angle FCE, [I 29.

and the angle ADE is equal to the angle EFC [I 29.

therefore AD=CF and DE=EF [I 26

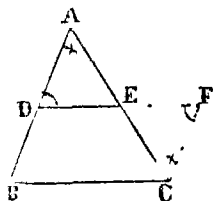
But AD=DB

∴ CF=BD, and they are parallel.

∴ DF is parallel and equal to BC [I 33.

But DE is half of DF

∴ DE is half of BC

Wherefore, *the straight line joining &c* Q E D

Proposition III Theorem

The straight line drawn from the middle point of a side of a triangle parallel to the base bisects the other side

Let BC be the base of the triangle ABC and D the middle point of AC DE is drawn parallel to CB meeting AB at the point E AB *is bisected at the point* E

Draw DF parallel to AB

Because DE is parallel to CB [*Cons* and AC falls upon them,

therefore the angle ADE is equal to the angle ACF, [I 29

and because DF is parallel to AB and CA falls upon them,

therefore the angle CAB is equal to the angle CDF. [I 29

also AD is equal to DC [*Hyp.*

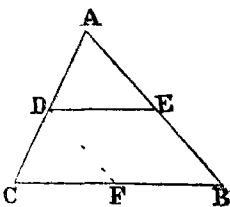
Therefore AE is equal to DF

Again, because DFBE is a parallelogram,

therefore DF is equal to EB [I 34

But AE has been proved equal to DF

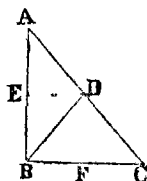
therefore AE is equal to EB

Wherefore, *the straight line drawn &c.* Q. E. D.

Proposition IV Theorem.

The straight line joining the right angle with the middle point of the hypotenuse of a right-angled triangle is equal to half the hypotenuse

Let ABC be a triangle, in which ABC is a right angle Let D be the middle point of AC DB *shall be equal to the half of* AC

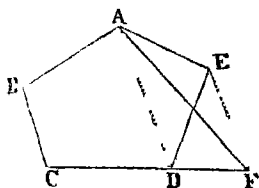
Draw DE parallel to BC and DF parallel to AB.

Because in the triangles AED, BED,
the side AE is equal to the side BE, (Prop III)
the side ED is common,
and the angle AED is equal to the angle BED (Ax 11)
Therefore AD=BD.
Likewise BD=DC
Therefore BD=AD=DC
=half of AC

Wherefore, *the straight line &c* Q. E D

Proposition V Problem

To construct a rectilineal figure equal to a given rectilineal figure and having the number of its sides one less than that of the given figure and thence to construct a triangle equal to a given rectilineal figure

Let ABCDE be a rectilineal figure, to construct a rectilineal figure equal to ABCDE and having one side less than the sides of ABCDE

Join AD, from E draw EF parallel to AD meeting CD produced at F Join AF ABCF is *the required figure*

Because EF is parallel to AD,
the triangle AFD is equal to the triangle AED [I. 37
Add to these equals the figure ABCD,
therefore the whole figure ABCF is equal to the whole figure ABCDE.

Wherefore, ABCF *is the required figure*

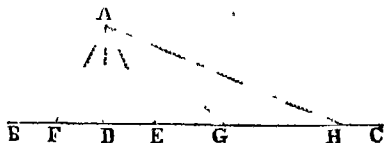
Because, this process can be repeated any number of times, any polygon can be reduced to a triangle equal to the polygon in area. Q E F

Maxima and Minima.

Def *A constant magnitude is that which retains the same value though its position varies according to a given law*

Def *When a geometrical magnitude, (a line, an angle or an area) subject to some given conditions, increases continuously for some time and then begins to decrease, it is a maximum at the end of the increase, also, if it decreases continuously for some time and then begins to increase, it is a minimum at the end of the decrease*

Proposition VI Theorem

Of all straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the least, and that which makes a greater angle with the perpendicular is greater than that which makes a less angle, and those which make equal angles with the perpendicular are equal

Let A be the given point and BC the given straight line

Draw AD perpendicular to BC Draw other lines AH, AG, AE so that the angle DAH is greater than the angle DAG, and the angle DAG greater than the angle DAE Also make the angle DAF equal to the angle DAE

Then AD *is the least* AH *is greater than* AG, AG *greater than* AE, AE *greater than* AD, *also* AF *is equal to* AE

Because ADE is a right angle, ∴ AED is an acute angle

∴ the side AE is greater than the side AD [I 19

The angle AEG is obtuse, for it is greater than ADE.

∴ the angle AEG is greater than the angle AGE

∴ the side AG is greater than AE [I 19.

Likewise, AH is greater than AG

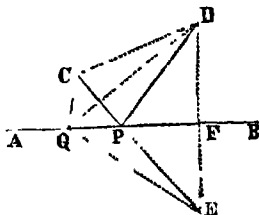
Again in the triangles DAF, DAE,

$\angle$FAD $= \angle$EAD, $\angle$FDA $= \angle$EDA (Ax 11) and AD is common,

∴ AF=AE

Wherefore, *of all straight lines &c* Q E D.

Proposition VII Problem

Given a straight line and two points on one side of it, to find a point in the given straight line so that the sum of its distances from the given points is the least possible.

Let AB be the given straight line and C, D the given points To find a point in AB so that the sum of its distances from C and D may be the least possible

Draw DF perpendicular to AB and produce DF to E making FE=DF Join CE cutting AB at P P *is the required point*

Take any point Q in AB

Join CQ, DQ, EQ and PD

Because, in the triangles DPF, EPF,

the side DF= the side EF, PF is common and the

$\angle$PFD=$\angle$PFE (Ax 11), $\therefore$ PD=PE [I 4.

Similarly, DQ=EQ

To the equals PD, PE add CP,

then the sum of CP, PD=the sum of CP, PE, that is, CE

Likewise the sum of CQ, DQ=sum of CQ, QE

But the sum of CQ, QE is greater than CE (I 20)

$\therefore$ the sum of CQ, QE is greater than the sum of CP, PD

$\therefore$ the sum of CQ, DQ is greater than the sum of CP, PD

Likewise if any other point be taken in AB, we can shew that the sum of its distances from the given points is greater than the sum of CP, DP

Therefore, P is the required point

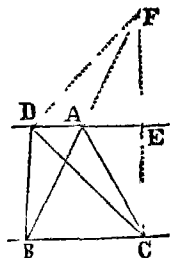
Wherefore, *in the straight line* AB *a point* P *is found so that the sum of its distances from* C, D *is the least possible.* Q E. F.

Proposition VIII Theorem

Of all triangles on the same base and between the same parallels, the perimeter is the least possible when the triangle is isosceles

Let the isosceles triangle ABC and any other triangle DBC be on the same base BC and between the same parallels DA, BC

The perimeter of ABC *is the least possible*

Draw CE at right angles to BC meeting DA produced at E

Let BA produced meet CE produced at F Join DF

The ∠BCF = the sum of the ∠s CBF and BFC, but ∠ABC = ∠ACB (I 5), ∴ ∠ACF = ∠AFC

∴ AF = AC (I 6) = AB

∴ EF = EC [Prop III

The angle DEF = the angle BCF [I 29.

= a right angle

Because in the triangles DEC and DEF,

EC = EF, DE is common,
and the angle DEC = the angle DEF [Ax 11

∴ DF = DC [I. 4.

∴ the sum of BD, DC = sum of BD, DF, which is greater than BF [I 20.

But BF = sum of BA, AC

∴ sum of BD, DC is greater than the sum of BA, AC

∴ the sum of BD, DC, BC is greater than the sum of BA AC, BC

If there be any other triangle on the same base BC and between the same parallels DA, BC, then its perimeter will be greater than that of ABC

∴ the perimeter of ABC is the least possible. Q E. D.

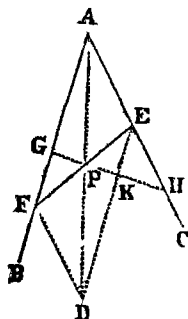
Proposition IX Problem

Given an angle and a point within it, to draw a straight line through the given point which will make a triangle of least possible area with the arms of the given angle

Let BAC be the angle and P the given point A straight line is to be drawn through P which will make a triangle of least possible area with AB, AC

Join AP, and produce it to D making PD equal to AP

From D draw DE parallel to BA meeting AC at E Join EP and produce EP to meet AB at F

AEF *is the required triangle*
Join FD

Through P draw any other line GPH meeting AB at G and AC at H, and *cutting ED at K*

To shew that the triangle AFE is less than the triangle AGH

In the triangles APF, DPE

$$AP=PD\,,$$
$$\angle PAF=\angle PDE, \qquad (I\ 29)$$
$$\text{and } \angle APF=\angle DPE, \qquad (I\ 15)$$
$$\therefore\ PF=PE \qquad (I\ 26)$$

Again in the triangles GPF, KPE,

$$\left.\begin{array}{ll} FP=PE, & \\ \angle PGF=\angle PKE, & (I\ 29) \\ \text{and } \angle GPF=\angle KPE\,, & (I\ 15) \end{array}\right\}$$
$$\therefore\ \triangle GPF=\triangle KPE \qquad (I\ 26)$$

Add to these equals the figure AGPE

∴ triangle AFE=quadrilateral AGKE.

∴ triangle AFE is less than the triangle AGH.

Wherefore, *the triangle* AFE *is the least possible* Q E F

COR The base of the least possible triangle is bisected at the given point

Loci.

Def. If any and every point on a line, or group of lines (straight or curved) satisfies an assigned condition, and no other point does so, then that line, or group of lines, is called the locus of the point, satisfying that condition.

Proposition X. Theorem.

The locus of a point at a given distance from a given point is a circle

Let A be the given point and B the given distance

With A as centre and B as radius describe the circle CF The distance of every point on the circumference of the circle CF from A is equal to B

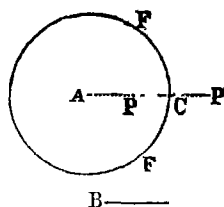
The distance of any other point which is not on the circumference is either greater or less than B

Take any point P which is not on the circumference Join AP and let AP or AP produced cut the circumference in C

AP is unequal to AC, but AC is equal to B

Therefore AP is unequal to B

Therefore the distance of P from A is not equal to B

Hence the circumference of the circle CF is the locus of the point whose distance from A is equal to the given distance B

Proposition XI Problem

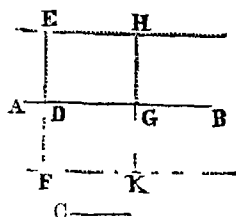
To find the locus of a point at a given distance from a given straight line

Let AB be the given straight line, C the given distance, *it is required to find the locus of a point at the distance C from* AB

Let E and H be points on the same side of AB, such that perpendiculars ED, HG on AB are each of them equal to C Join EH

∵ ED and HG are each of them perpendicular to AB,

∴ ED is parallel to HG [I 28

But ED=HG, ∴ EH is parallel to AB [I 33.

Similarly, if F and K, points on the other side of AB, be taken, such that the perpendiculars FD, KG on AB be equal to C, then FK is parallel to AB.

∴ the distance of every point in EH or FK is equal to C.

∴ EH and FK form the required locus

If any point be taken outside EH or FK, we can shew that its distance from AB is not equal to C

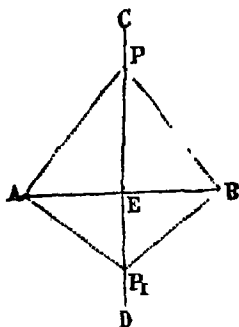
∴ the locus of a point at a given distance C from AB is the pair of parallel lines EH and FK Q E F

Proposition XII Problem.

To find the locus of a point equidistant from the extremities of a given finite straight line

Let AB be the given finite straight line *It is required to find the locus of a point equidistant from* A, B.

Let P be any point such that AP is equal to BP

Bisect AB at E Then E is equidistant from A, B Join PE and produce it to C, D *Then* CD *is the locus of the point equidistant from* A *and* B

In the triangles AEP, BEP,

because AE is equal to BE, [*Constr.*

EP is common,

and AP=BP,

∴ the angle AEP is equal to the angle BEP, [I 8

∴ PE is at right angles to AB

If any point P_1 be taken in CD,

AP_1 will be equal to BP_1 [I 4.

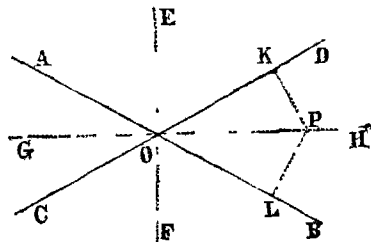
Therefore every point in CD is equidistant from A, B

Therefore CD, which bisects AB at right angles, is the locus of the point equidistant from A, B. Q. E F.

Proposition XIII. Problem.

To find the locus of a point equidistant from two straight lines which intersect each other

Let AB, CD intersect at O. *It is required to find the locus of a point equidistant from* AB, CD

Let P be a point, such that the perpendicular PK on OD is equal to the perpendicular PL on OB Join OP

$$OP^2 = OK^2 + PK^2 = OL^2 + PL^2$$

But PK = PL, ∴ $PK^2 = PL^2$

∴ $OK^2 = OL^2$, ∴ OK = OL

By I 8, the angle KOP = the angle LOP

∴ P is a point on the bisector of the angle KOL

If any other point H be taken on OP, and perpendiculars be drawn from that point on OD and OL, we can prove by I 26 that those perpendiculars are equal

Hence OH is the locus of the point equidistant from OD, OB

Similarly, if the angle AOD be bisected by OE, OE is the locus of the point equidistant from OA, OD

If EO be produced to F and HO to G, OF will bisect the angle COB and GO will bisect the angle AOC

Hence every point in GO is equidistant from OA, OC and every point in OF is equidistant from OC, OB

Half of the sum of the angles AOD, DOB is a right angle

∴ the sum of the angles EOD and DOH, or the angle EOH is a right angle

∴ EF is at right angles to GH

Hence the locus of the point equidistant from AB, CD is the pair of straight lines EF, GH at right angles to one another, which bisect the angles made by the intersecting lines AB, CD Q E F

EXERCISES

1 Find the locus of the vertex of an isosceles triangle on a given base

2 Find the locus of a point at which two adjacent sides of a square subtend equal angles

3 Given the area of a triangle and the base, find the locus of the vertex

Intersection of Loci

If a locus of a point satisfying one condition be constructed and also a second locus satisfying a second condition, then the point or points which satisfy both conditions must be the point or points common to both loci and therefore must be the point or points where the loci intersect

Proposition XIV. Problem.

Given the base, a side and the altitude of a triangle, to construct it

Let AB be the given base, C the side from the extremity A, and D the altitude To construct a triangle with these data

From the point A draw AE at right angles to AB and make AE equal to C [I 3

That the problem be possible D must be not greater than C, or not greater than AE

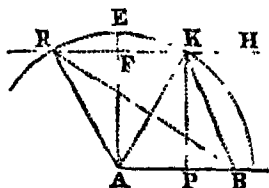
From AE cut off AF equal to D

Draw FH parallel to AB, and with A as centre and AE or C as radius describe the circle REK cutting FH at K and HF produced at R Join AK and KB, also join AR, RB AKB or ARB is the triangle with the required data

FH is the locus of the end of the altitude, that is, of the vertex [*Prop* XI

Also the circumference REK is the locus of the end of AE (which is equal to C), that is of the vertex

But K and R, the points of intersection of the two loci of the vertex, are the only two points which lie on FH as well as on the circumference of the circle REK

Therefore, each of AKB and ARB is the triangle with the given data Q E F.

EXERCISES

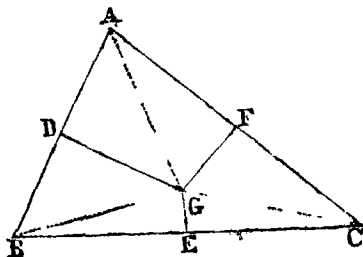
1 In a given straight line find a point equidistant from two given points

2 Describe an isosceles triangle on a given base, each of whose sides shall be double of the base.

3 Find a point at a given distance from a given point, and at the same distance from a given straight line

Concurrency.

Proposition XV. Theorem.

The three straight lines at right angles to the sides of a triangle at their middle points are concurrent, and their point of concurrence is equidistant from the three angles of the triangle.

Let ABC be a triangle, D, E, F the middle points of the sides AB, BC and CA respectively *The straight lines drawn from* D, E, F *at right angles to the sides meet at a point*

Draw EG and FG at right angles to the sides BC and CA respectively, meeting each other at G Join AG BG, CG

Every point in EG is equidistant from B and C [*Prop* XII.

∴ BG=CG

Also every point in FG is equidistant from A, C

∴ CG=AG [*Prop* XII.

∴ BG=CG=AG

Join DG

Because in the triangles ADG, BDG
the side AD=the side BD
the side DG is common,
and the base AG=the base BG,
therefore the angle ADG=the angle BDG [I 8.

Therefore DG is at right angles to AB [*Def* 11.

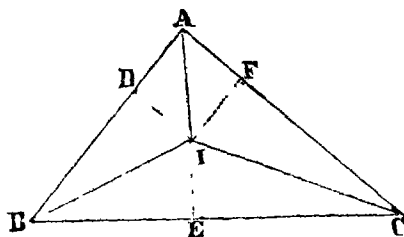
Therefore a line drawn from D at right angles to AB passes through G

AG, BG and CG have been proved to be equal, therefore G is equidistant from A, B, C Q E D

Def *The point which is equidistant from the three angles of a triangle is called the* **circum-centre** *of the triangle*

Note The circum centre of a triangle means the centre of the circle circumscribing the triangle

Proposition XVI. Theorem

The three lines bisecting the angles of a triangle are concurrent; and the point of their concurrence is equidistant from the sides.

Let ABC be a triangle, bisect the angles B and C by BI and CI, meeting at I, draw ID, IE and IF perpendiculars to AB, BC, CA respectively

Every point in BI is equidistant from AB and BC, [*Prop* XIII.

∴ DI=EI

Likewise, every point in CI is equidistant from BC, AC

∴ EI=FI

∴ DI=EI=FI

Because the square on AI=the squares on AD, DI, [I 47

and also the square on AI=the squares on AF, FI, [I 47

∴ the squares on AD, DI=the squares on AF, FI [*Ax* 1

But the square on DI=square on FI [I 46, Cor 2

Take away these equals,

then the square on AD=the square on AF

∴ AD=AF

DI=FI,

and the angle ADI=the angle AFI [*Ax*. 11

∴ the angle DAI=the angle FAI. [I 4

∴ the straight line bisecting the angle DAF passes through I.

∴ the three lines bisecting the angles A, B, C are concurrent

Also DI, EI and FI have been proved to be equal;

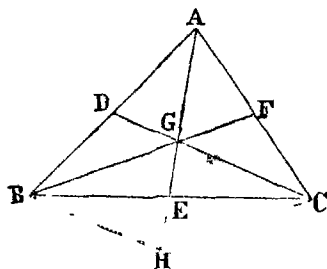
∴ I is equidistant from the sides

Q E D.

Def. *The point of concurrence of the bisectors of the angles of a triangle is called the* ***in-centre*** *of the triangle*

NOTE In-centre of a triangle is an abbreviation for the centre of the circle inscribed in the triangle

Proposition XVII Theorem

The three medians of a triangle are concurrent, and the distance of their point of concurrence from any angle is two-thirds of the median along which it is measured

Let ABC be a triangle, let BF and CD be two medians intersecting at G Join AG and produce it to cut BC at E and the parallel to DC through B at H Join CH

Because D is the middle point of AB, and DG is parallel to BH, therefore AG is equal to GH [*Prop* III

Because G, F are the middle points of AH and AC,

therefore GF is parallel to HC [*Prop* II

∴ BHCG is a parallelogram

∴ the diagonals GH and BC bisect each other at E [I 34, *Ex* 1.

∴ BE is equal to EC

∴ E is the middle point of BC

The straight line joining A, E passes through G

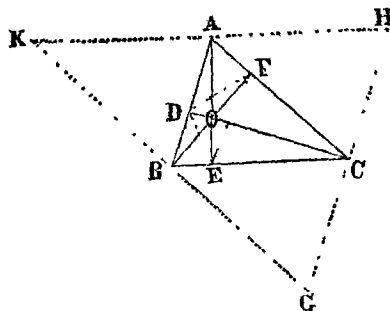
∴ the three medians pass through G

Also AG=GH=2 GE

Likewise we can prove BG=2 GF and CG=2 DG Q E D

Def *The point of concurrence of the medians of a triangle is called its* ***centroid***

Proposition XVIII. Theorem.

The three altitudes of a triangle are concurrent

Let ABC be a triangle, through A, B, C draw KAH, KBG, GCH parallel to BC, AC, AB respectively, forming the triangle GHK

ABCH, ABGC and KBCA are parallelograms

∴ KA=BC=AH [I 34

Likewise, KB=BG and GC=CH

Therefore A, B, C are the middle points of the sides of the triangle GHK

Perpendiculars at the middle points of the sides of GHK are concurrent [*Prop* XV

Let O be the point of concurrence

Produce AO, BO, CO to meet BC, CA and AB respectively at E F and D.

∵ KH is parallel to BC and EA is perpendicular to KH,

∴ AE is perpendicular to BC, since the sum of the angles HAE and AEC is equal to two right angles [I 29

Likewise, CD and BF are perpendiculars to AB and AC respectively

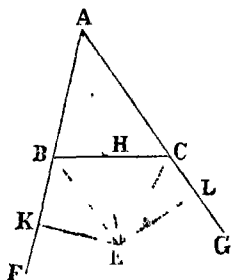
Therefore, the three altitudes of the triangle ABC are concurrent.

Q E D.

Def *The point of concurrence of the three altitudes of a triangle is called the* **ortho-centre** *of the triangle*

Obs The centroid, circum centre, in centre and ortho-centre of an equilateral triangle coincide in one point, which therefore may be called the centre of the triangle

Proposition XIX. Theorem

If any two sides of a triangle be produced through the extremities of the base, the bisectors of the exterior angles are concurrent with the bisector of the vertical angle, the point of concurrence is equidistant from the two sides produced and the base

Let ABC be a triangle Produce AB, AC, to F and G let BE, CE bisecting the angles CBF and BCG respectively meet at E

Draw EH, EK, EL perpendiculars to BC, AF and AG respectively

Because BE is the locus of the point equidistant from BC and BF, and because CE is the locus of the point equidistant from BC and CG, (Prop XIII), therefore E, the intersection of BE and CE, is equidistant from BC, BF and CG ∴ KE=EH=EL

As in Prop XVI, we can prove that AE bisects the angle BAC

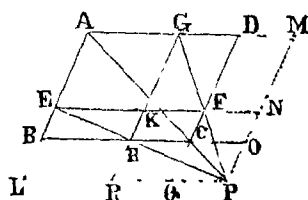
Therefore the straight line bisecting the angle BAC passes through E

Wherefore, *if any two sides of a triangle* &c Q E D.

Def *The point which is equidistant from the two sides produced and the third side of a triangle is called its* ***ex-centre***

NOTE If a circle be described with E as centre and EK EH or EL as radius, the circle is called an escribed circle of the triangle, the ex-centre means the centre of the escribed circle

Proposition XX Theorem.

If two parallels be drawn to two adjacent sides of a parallelogram so as to divide it into four parallelograms, the diagonals of two parallelograms at any two opposite corners of the original parallelogram (the diagonals not passing through the corners) and the diagonal of the original parallelogram passing through its other corners are concurrent.

Let EF, GH be two parallels to the sides AD, AB of the parallelogram ABCD *Then* EH, AC GF, *or* EG, HF, BD *are concurrent*

Produce EH, GF to meet at P Through P draw PM, PL parallels to AB, AD Let AB, GH, DC produced meet PL at L R, Q, and let AD, EF, BC produced meet PM at M, N, O

∵ complement RF=complement FM,

∴ ▭RF+▭FO=▭FM+▭FO=▭CM.

And complement HN=complement LH

∴ ▭HN+▭HQ=▭LH+▭HQ=▭LC

But ▭RF+▭FO=▭HN+▭HQ

∴ ▭CM=▭LC

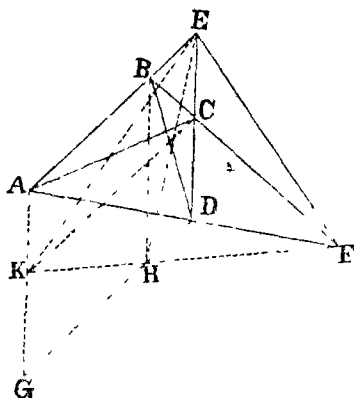
Hence the point C must be on the diagonal AP [I 43, Ex 3.

Therefore, the diagonal AC produced must pass through P

Q E D.

Def *If each pair of opposite sides in a quadrilateral be produced to meet, the straight line joining the points of intersection is called its* **third diagonal,** *and the figure thus formed is called a* **complete quadrilateral**

Proposition XXI. Theorem

The middle points of the three diagonals of a complete quadrilateral are collinear

Let ABCD be a quadrilateral, produce AB, DC to meet at E and produce BC, AD, to meet at F *The middle points of* AC BD, EF *are collinear*

Draw DG, CK parallel to AE, and AG, BH parallel to ED Produce KH

BC, AD, KH are concurrent [*Prop* XX

Join EK, EH

The middle points of EK, EH, EF are collinear

But the middle points of EK, EH are also the middle points of AC, BD respectively [I 34, Ex. 1.

Hence the middle points of AC, BD, EF are collinear

Wherefore, *the middle points of* &c. Q. E. D.

GEOMETRICAL EXERCISES ON BOOK I.

ON GEOMETRICAL ANALYSIS

The word **analysis** (Gr from *ana*, again and *lusis*, *luo*, to loosen) in general, means resolution of anything into its constituent or original elements The word **synthesis** (Gr *sunthesis* from *sun*, with and *thesis*, a placing) generally means composition or the putting of two or more things together In Geometry these words are used in a more restricted sense In analysis the quæsitum of a problem, or *the conclusion of a theorem, is assumed to be effected or admitted*, and from the assumption, consequences are traced by a series of geometrical constructions, and reasonings, till they terminate in the data of the problem or in the hypothesis of the theorem In other words, geometrical analysis is an attempt to trace backwards step by step from the quæsitum (or the conclusion) to some original condition (or principle) which shows the necessary construction (or which is known to be true) It thus shows on what fundamental truth the quæsitum or the conclusion depends On the other hand in synthesis we start with the data or the hypothesis, and by a chain of reasoning or by construction (or by both) arrive at the quæsitum or conclusion, aided by definitions postulates, axioms &c

The propositions in the Elements of Euclid are proved by the synthetical method By this method Euclid begins with certain principles or truths already obtained or originally admitted (for instance definitions, postulates, axioms or earlier propositions) and proceeds by deducing successive inferences from them to the solution of problems or to the demonstration of theorems But although Euclid employed this method in the working of his propositions, we do not know what suggested to him to draw those particular inferences in order to arrive at the required result And as a large number of inferences may be drawn in some cases, we may adopt the wrong inferences and thus not attain what we desire Hence the better method is to proceed analytically It may be well illustrated in the solution of geometrical exercises The number of such exercises is practically indefinite For, a vast number of combinations may be made of the principles established in geometry, moreover, there are many other geometrical principles as yet unknown to us, as is seen in the recent discoveries of French Geometricians.

The utility of analysis lies in the fact that it enables us to search for a clue to the solution of a proposition For it gradually narrows down the area within which must lie the possible ways of solving it Of course analysis alone is incomplete, for it does not actually solve a proposition but merely shows us *how* to do it We must employ synthesis, when we have once obtained the clue we were seeking, to make the right constructions from the data, or to draw

the right inferences from the hypothesis, in order to arrive at the required quæsitum or conclusion Hence the method of procedure is in general the following —

Start with the quæsitum (or the conclusion) and work backwards step by step till you arrive at the original principle on which it depends, that is, till you obtain the clue you were seeking Now *adopt the inverse process, in other words reverse all the steps in* the analysis, starting from the clue, that is, the data or the hypothesis, till you arrive at the desired quæsitum or conclusion, of course at each point the right inferences must be drawn as indicated by the successive steps of the analysis when arranged in the reverse order

We give the following directions for the analysis of problems and theorems

ANALYSIS OF A PROBLEM

1 Assume that the quæsitum has been obtained by drawing the required diagram

2 Examine the relations of the points, lines, angles, triangles &c, in the diagram, and find out whether the assumed quæsitum depends on some theorem or problem in the Elements

3 In case of failure, produce the lines, draw other lines whether perpendicular or parallel, join given points, or points assumed, and describe circles if need be then proceed on to find out the dependence of the assumed quæsitum on some theorem or problem already proved or solved

ANALYSIS OF A THEOREM

1 Assume that the conclusion is true

2 Examine the diagram, and draw other lines, etc, if need be, and find out the relation of the assumed conclusion to any truth respecting the diagram which has been already proved

NOTE There is no fixed rule however as to *what* particular lines, etc we must draw To be successful at the first attempt we must have realized the principles of the Elements But the student would do well to employ *all* the data (or the hypothesis) successively in making the constructions, thus gradually narrowing down the limits within which must lie the right construction

OBS It is not always necessary to go through the whole process of analysis and synthesis For, when certain inferences are obvious from our data or hypothesis, we might start from them as if they were the original data or hypothesis, thereby shortening our labour

Exercises Solved.

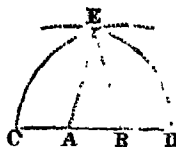
Exercise 1. Problem

On a given straight line to construct an isosceles triangle having each of the equal sides double of the base

Let AB be the given straight line, it is required to describe an isosceles triangle on AB, so that the equal sides shall be double of AB

ANALYSIS

Let AEB be the isosceles triangle required, so that AE or BE is double of AB

Now if we follow the construction in Prop 1 in the Elements, we see that the circle with A as centre and AE as radius will cut AB produced at D, so that AD=AE =2AB Similarly the circle with B as centre and BE as radius will cut BA produced at C, so that BC=2BA Hence CA=AB=BD This gives us the clue we were seeking

SYNTHESIS

Produce AB both ways and cut off AC and BD making each of them equal to AB [I 3

With A as centre and AD as radius describe the circle DE, and with B as centre and BC as radius describe the circle CE cutting the other circle at the point E [*Post* 3.

Join AE, BE

Then AEB *is the required triangle*

Because A is the centre of the circle DE,

therefore AE is equal to AD,

but AD is double of AB (since AB is equal to BD),

therefore AE is double of AB

Likewise, we can prove that BE is double of AB.

Therefore AE and BE are each of them double of AB

Hence AE is equal to BE

Wherefore, ABE is the required triangle Q. E. F.

Exercise 2 Problem

To trisect a right angle.

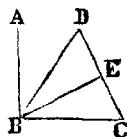
Let ABC be a right angle; it is required to trisect it

ANALYSIS

Suppose BD, BE trisect the angle ABC
Then the angles ABD, DBE, EBC are equal

Therefore DBC is equal to two-thirds of a right angle

But we know that the angle of an equilateral triangle is two-thirds of a right angle, and if we bisect this angle, each part is equal to one-third of a right angle

Hence we adopt the following synthesis —

SYNTHESIS

On BC describe the equilateral triangle DBC Bisect the angle DBC by BE

Then the angle ABC *is trisected by* BD BE

Because the angle DBC is equal to two-thirds of a right angle, [I 32 *Cor*

therefore the angle ABD is one third of a right angle

Also each of the angles DBE, EBC is equal to one-third of a right angle

Therefore the angles ABD DBE and EBC are equal

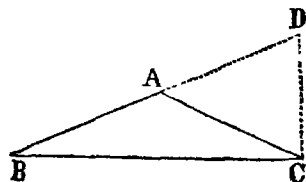
Wherefore, the angle ABC is trisected by BD, BE Q E F

Exercise 3 Problem

Given the base one of the angles at the base, and the sum of the two remaining sides of a triangle to construct it.

ANALYSIS

Suppose BAC be the triangle required, whose base is BC, the given angle is the angle at B, and the sum of the sides equal to BD

Because BA, AC are equal to BD, take away the common part BA, then AD is equal to AC [*Ax* 3

Hence we know that the vertex of the triangle lies on BD, and is equidistant from C and D

Therefore the angle ADC is equal to the angle ACD. [I. 5.

SYNTHESIS

Let BC be the given base, DBC the given angle, and BD equal to the sum of the sides

Join DC; at the point C and in the straight line DC make the angle DCA equal to the angle BDC, CA meeting BD at the point A [I. 23

ABC *is the triangle required.*

Because the angle ACD is equal to the angle ADC,
therefore AD is equal to AC [I 6

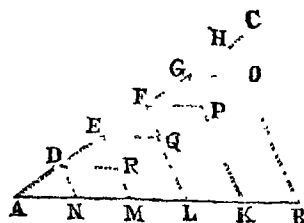
Add to these equals BA

Then BA, AC are equal to BA, AD, that is, to BD [*Ax* 2

Therefore of the triangle ABC, BC is the given base, the angle ABC the given angle, and the sum of BA, AC is equal to the given sum

Wherefore, ABC is the required triangle Q E F

Exercise 4 Problem

To divide a straight line into any number of equal parts

Let AB be a given straight line, to divide AB into any number of equal parts—*say five equal parts*

From any extremity A draw a line AC making an angle BAC From AC cut off five equal parts AD, DE, EF, FG, GH Join HB

From D, E, F, G, draw lines DN EM, FL, GK parallel to HB From D, E, F, G, draw DR, EQ, FP, GO parallel to AB cutting EM, FL, GK, HB at R, Q, P, O respectively

Because in the triangles DAN, EDR, FEQ, GFP, HGO,
the angles DAN, EDR, FEQ, GFP, HGO are equal, [I 29
and the angles ADN, DER EFQ, FGP, GHO are equal, [I 29,
and the sides AD, DE, EF, FG, GH are equal

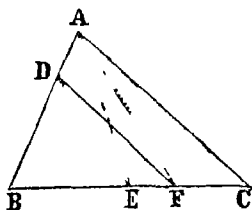
∴ AN=DR=EQ=FP=GO [I 26

But DR, EQ, FP, and GO are respectively equal
to NM, ML, LK and KB [I 34.

∴ AN=NM=ML=LK=KB

∴ AB is divided into five equal parts. Q. E F

Exercise 5 Problem.

To bisect a triangle by a straight line drawn from a given point in one of its sides

ANALYSIS

Let ABC be a triangle and D a point in AB To bisect the triangle ABC by drawing a line from D

Let DF bisect the triangle

Now we know that if E be the middle point of BC, AE will bisect the triangle [I 38

Hence, joining DE, and AF, we see that the triangle DBF =triangle ABE ∴ triangle DFE=triangle DAE

∴ AF is parallel to DE [I 39.

Thus we obtain the clue to the following synthesis —

SYNTHESIS.

Bisect BC at E, join DE, and draw AF parallel to DE

DF *will bisect the triangle* ABC.

Join AE

△DAE=△DFE [I. 37.

To each add △DBE

∴ △ABE=△DBF

But △ABE=half of △ABC [I 38.

∴ △DBF=half of △ABC.

Therefore, DF bisects the triangle ABC. Q E F

MISCELLANEOUS EXERCISES ON BOOK I.

1 In the figure, Euc I 1, if AB be produced both ways to meet the circles in D, E, and DC, EC be joined, then the triangle CDE is isosceles

2 To find a point in a given straight line so that straight lines drawn from it to two given points without it will make equal angles with the given straight line

3 If the base of an isosceles triangle be produced both ways, the exterior angles which it makes with the equal sides shall be equal

4 The straight lines bisecting the angles at the base of an isosceles triangle, together with the base, make up an isosceles triangle

5 The two sides of a triangle are produced, if the exterior angles be equal, prove that the triangle is isosceles

6 If from the middle points of two sides of a triangle straight lines be drawn at right angles to them intersecting at a point, the straight line joining the point of intersection with the middle point of the third side will be at right angles to that side

7 To describe a circle which shall pass through three given points not in a straight line

8 ABC is a triangle, find a point D in BA, or BA produced, so that BD may be equal to CD

9 To find two points in a straight line which will be equidistant from a given point without the given straight line

10 If two straight lines cut one another, and the vertical angles be bisected, the bisecting lines are in one and the same straight line

11 To construct a triangle when two sides and an angle opposite to one of them are given

12 To construct a triangle, having given one side, an adjacent angle, and the difference of the other two sides

13 *The opposite angles of a rhombus are equal, also they are* bisected by the diagonals

14 To find a point in a given straight line so that straight lines drawn from it to two given points shall include an angle which will be bisected by the given straight line In what cases does the construction fail?

15 Given one side, and the perpendicular on the hypotenuse of a right angled triangle to construct it

16 One side of a triangle is greater than, equal to, or less than another, according as the angle opposite to the former, is greater than, equal to, or less than the angle opposite to the latter

17 The three sides of a triangle taken together are greater than the double of any one side, but less than the double of any two sides

18 In an isosceles triangle, if a straight line be drawn from the angle opposite the base, bisecting the angle, it bisects the base or if it bisect the base, it bisects the angle, and in either case, it cuts the base at right angles

19 The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other

20 Through a given point to draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other

21 Through a given point to draw a straight line such that the segments, intercepted by perpendiculars let fall upon it from two given points, shall be equal

22 Given one side of a right angled triangle and the sum or difference between the hypotenuse and the other side construct it

23 Two lines making a fixed angle between them revolve round the angular point and cut a line extended both ways beneath the angular point, of all the triangles thus formed, that is the least whose vertical angle is bisected by the perpendicular from the angular point upon the extended line, and of the rest that which is nearer to the perpendicular is less than one more remote, and only two equal triangles can be formed one on each side of the perpendicular

24 If two points be taken in the base of a triangle at equal distances from, and on opposite sides of the middle point of the base, show that the two lines joining these points with the vertex of the triangle, shall always be less than the sides of the triangle, but greater than twice the line joining the middle of the base with the vertex

25 ABC is an isosceles triangle The line AD bisecting BC is produced to E, and DE made equal to AD, E is joined to the middle points of AB and AC by lines cutting BC in F and G. Shew that AFEG is a rhombus

26 To draw from a given point three straight lines equal to three given lines not all equal, so that their extremities may lie in a straight line and the end of the least be equally distant from the ends of the others

27 Given in any triangle, one side, an angle at one extremity of it, and the length of the perpendicular drawn from its middle point to meet those drawn from the middle points of the other two sides to construct the triangle

28 Given perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base construct the triangle

29 Find the locus of a point such that the difference of its distances from two given straight lines may be of constant length

30 Find the locus of a point at which two adjacent sides of a rectangle subtend supplementary angles

31 Find the locus of a point at which two equal intercepts on a given line subtend equal angles

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32 If a straight line falling on two other straight lines, make the two exterior angles on the same side of it equal to two right angles, these two straight lines are parallel

33 To find a point in the side AC of a triangle ABC from which a straight line may be drawn to another side AB parallel to the third side, and equal to that segment EB (of the side to which it is drawn) that is conterminous with the parallel side BC

34 If the diameters of a parallelogram are at right angles, the sides are all equal

35 If two straight lines bisect one another, the straight lines joining their extremities form a parallelogram

36 Draw a line DE parallel to the base BC of a triangle ABC, so that it will cut off a triangle ADE which would be equal to one fourth of ABC

37 Through a given point between two given straight lines, to draw a straight line to meet them and to be bisected at the given point

38 The straight lines joining successively the three middle points of the sides of a triangle divide the triangle into four triangles which are identically equal

39 A trapezoid is equal to a rectangle whose base is half the sum of the two parallel sides, and whose altitude is the perpendicular distance between them

40 In a triangle if a straight line be drawn from the vertical angle perpendicular to the base and also a bisector be drawn of the same angle, then the angle between these two lines is equal to the difference of the base-angles

41 Construct a triangle when two angles and a side opposite to one of them are given

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42 If from the extremities of the base of a triangle, two segments be cut off, each equal to its adjacent side, and straight lines be drawn from the vertex to the points of section, these straight lines will contain an angle equal to half the sum of the angles at the base of the triangle.

43 If three straight lines be drawn making equal angles with the three sides of a triangle towards the same parts, they will form a triangle equiangular to the given triangle

44 If four points be taken on the sides of a rhombus at equal distances from the angular points, the figure which is formed by joining them is a parallelogram

45 ABC is a triangle, and the exterior angles formed by producing AC both ways are bisected by straight lines that meet in D shew that the angle ADC and half the angle ABC make up a right angle

46 Prove that the angle intercepted between the perpendicular on the hypotenuse and the line which bisects the right angle is equal to half the difference of the base angles

47 One of the acute angles of a right angled triangle is three times as great as the other trisect the larger of these

48 On the sides of an acute-angled triangle, isosceles triangles are described externally determine the nature of the equal angles of these isosceles triangles in order that the sides of these triangles terminated at each angular point of the given triangle, may be in one and the same straight line

49 Given an isosceles triangle, to construct another on the same base with it, and having double its vertical angle When is this impossible?

50 Given the hypotenuse of a right angled triangle and the difference of the segments of the hypotenuse into which it is cut by a perpendicular on it from the right angle construct it

51 Given the perpendicular in an equilateral triangle - construct it.

52 Given one side and the sum of the other side and the hypotenuse of a right-angled triangle; construct it

53 Determine a point in the side of a triangle, from which straight lines drawn parallel to the other sides will form a rhombus.

53 Given the sum of the three sides; construct a triangle equiangular to a given one

55 If any point be taken in the side of an equilateral triangle and from it straight lines be drawn parallel to the other sides of the triangle, their sum is equal to a side of the triangle.

56 If the angle BAC of the triangle BAC be bisected by DA, and CA be produced to E making AE equal to AB, show that EB is parallel to AD

57 Given the difference between the hypotenuse and a side of an isosceles right angled triangle construct it.

58 To inscribe a square in an equilateral triangle

59 Given the perimeter of an isosceles right-angled triangle; to construct it

60 Given of a right-angled triangle, the sum of the hypotenuse and one side, and the contained angle to construct it.

61 Given of any triangle the base the difference of the sides and the vertical angle to construct it

62 To inscribe a square in a given triangle

63 To construct an isosceles triangle, whose vertical angle shall be four times one of its base angles

64 In a given square, inscribe an equilateral triangle one of whose angular points shall be in an angular point of the given square

65 Given of any triangle the base, the sum of the sides, and the vertical angle to construct it

66 If the angles at the base of an equilateral triangle be bisected and straight lines be drawn from the intersection of the bisecting lines parallel to the other two sides these lines will trisect the base

67 The difference of the angles at the base of any triangle is double of the angle between two lines drawn from the vertex, one bisecting the vertical angle and the other perpendicular to the base.

68 To draw a straight line which will pass through a given point and make equal angles with two given intersecting lines

69 The quadrilateral whose opposite sides are equal is a parallelogram

70 The quadrilateral whose opposite angles are equal is a parallelogram

71 The diagonals of a rectangle are equal

72 Through the point A of any triangle ABC a straight line is drawn parallel to the side CB, and through C a straight line CD of length equal to AB is drawn, meeting the straight line through A parallel to CB at D, and cutting AB at E Prove that CE is equal to EB, and AE to ED

73 From the extremities of the straight line AB perpendiculars AC, BD are drawn on opposite sides of it, such that AC and BD together are equal to AB Show that the straight line CD always makes the same angle with AB

74 Draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines

75 Find a point in the base of a triangle, from which lines drawn to each side of the triangle, and parallel to the other shall be equal.

76 Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram

77 Draw from a given point three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line

78 If from any point in the base of an isosceles triangle, two straight lines be drawn, making equal angles with the base, and terminated by the opposite sides, their sum is the same whatever point be taken

79. If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal

80 Determine a point in the side of any triangle, from which straight lines drawn parallel to the other sides of the triangle will form a rhombus

81. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

82 Find a point in a side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity

83 If in the diagonal of a parallelogram any two points equidistant from the extremities be joined with the opposite angles, a figure will be formed which is also a parallelogram

84 Three straight lines issue from a point draw another straight line cutting them so that the two segments of it intercepted between them may be equal to one another

85 A straight line AB is bisected in the point C prove that the sum of the perpendiculars from A and B on any straight line which does not pass between A and B is double the perpendicular from C on the same straight line

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86 Given the base of a triangle, the altitude and the median to the base to construct it

87 In the triangles ABC, DEF the sides AB, BC are respectively equal to the sides DE, EF, and the angle ABC is the supplement of the angle DEF prove that the areas of the two triangles are equal

88 From the middle point of the base AB of a triangle, a line is drawn cutting the sides BC and AC, produced if necessary, in L and M From C, CE is drawn parallel to LM to meet the base, produced if necessary at E Show that the triangle AME and BEL are equal

89 If two triangles stand on the same base and between the same parallels, and if a line be drawn parallel to the base, the smaller triangles cut off from the larger ones are also equal

90 If two triangles stand on the same base and on the same side of it, and if the middle points of the sides be joined, prove that a parallelogram will be formed by the joining lines

91 Given a triangle ABC and a point D in AB construct another triangle ADE equal to the former and having the common angle A

92 Through E, the bisection of the diagonal BD of a quadrilateral ABCD, draw FEG parallel to AC, and shew that AG will bisect the figure.

93 If from a point without a parallelogram straight lines be drawn to the extremities of two adjacent sides and also to the extremities of the diagonal passing through the angle included by them, of the triangles thus formed, that whose base is the diagonal is equal to the sum of the other two.

94 Given the middle points of the sides of a triangle, construct the triangle.

95 Bisect a given quadrilateral by a straight line drawn through one of its angular points.

96 Given the base, to find the locus of the vertices of an infinite number of triangles equal in area.

97 Given of any triangle the base and the lines which drawn from the extremities of the base bisect the other two sides, to construct it.

98 If two triangles have their areas equal and one side and an adjacent angle equal in each, they shall be equal in all respects.

99 Through a given point P on one side BC produced of a parallelogram, draw a straight line cutting the sides CD, DA in E and F, and AB produced in H, so that the triangle FDE may be equal to the sum of the triangles PCE and AFH.

100 ABCD is a square, AC its diagonal. Bisect AD in E, join BE cutting AC in F, then shall the triangle AFE be equal to half of the triangle CEF, to one third of the triangle ABE and to one fourth of the triangle BCF.

101 ABCD is a parallelogram, and the straight line PQ, cutting BC in P and AD in Q, bisects it. Prove that the triangle PBQ is equal to the triangle PQD.

102 Within any triangle find a point such that the straight lines which join it with the angular points of the triangle divide the triangle into three equal parts.

103 Upon a given base describe an isosceles triangle equal to a given triangle.

104 Find the locus of a point such that the sum of its distances from two intersecting unlimited straight lines may be equal to a given straight line.

105 The locus of the middle points of a straight line drawn from a given point to a given straight line of unlimited length, is the straight line which bisects at right angles the perpendicular from the given point upon the given straight line

106 To construct a square double of a given square.

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107 Given the base, find the locus of the vertex of the triangle, the difference of the squares of whose sides is constant

108 Let ACB, ADB be two right-angled triangles having a common hypotenuse, join CD, and on CD produced both ways, draw perpendiculars AE, BF Show that the squares on CE, CF are together equal to the squares on DE, DF

109 In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn one to bisect the opposite side, and the other perpendicular to that side, these three straight lines divide the right angle into three equal parts

110 In the figure, Euc I 47, if DF, EK be joined, the sum of the angles at the bases of the triangles BFD, CEK is equal to one right angle

111 In fig Euc I 47, if BG and CH be joined, these lines will be parallel

112 In fig Euc I 47 if DB EC be produced to meet FG and KH in M N the triangles BFM, CKN are equiangular and equal to the triangle ABC

113 In fig Euc I 47, the sum of the squares on GH, KE, and FD will be equal to six times the square on the hypotenuse

114 In fig Euc I 47, produce FG, KH to meet in M, join MB MC and produce MA to cut BC in L prove that ML is perpendicular to BC

115 The perpendiculars from the angular points of a triangle on the opposite sides meet all in one point

116 If one of the acute angles of a right-angled triangle is double the other, prove that the equilateral triangle described on the hypotenuse is equal to the equilateral triangles described on the sides

117. In any triangle, if squares be described on the base, and on the other two sides, and if the perpendiculars to these sides, drawn from the extremities of the base, be produced to meet the opposite sides of the squares or those sides produced, the two rectangles cut off between these perpendiculars and the sides of the squares drawn from the extremities of the base, are together equal to the square on the base

118 The point of intersection of the straight lines bisecting two exterior angles of a triangle is joined with the opposite angle prove that the joining line bisects the angle

HINTS FOR SOLUTION.

BOOK I.

Prop 1.

1 The demonstration is the same as in the Proposition

2 See **Ex 1**, page 109

Prop 2

Let AB be the smaller of the two given straight lines AB, CD From the points A B draw AF, BF each equal to CD From the centre A at the distance AE describe the circle EGH, from the centre B at the distance BF describe the circle FGK cutting the circle EGH at G Join AG, BG ABG is the required triangle

Prop 4

1 Proceed as in the Proposition

2 Let ABC and DEF be two equal triangles Let BC be equal to EF and the angle ABC equal to the angle DEF If BA be not equal to ED, from ED cut off EG equal to BA The triangle ABC will be equal to the triangle GEF, etc

Prop 5

2 Let the diagonals of the rhombus $ABCD$ cut each other at O $AB=BC$, $\angle BAC=\angle BCA$ Similarly $\angle DAC=\angle DCA$. $\angle BAD=\angle BCD$ Now, of the two $\triangle$s BAD and BCD, the two sides BA, AD are equal to BC, CD respectively and $\angle BAD=\angle BCD$, $\therefore \triangle BAD=\triangle BCD$, and $\angle ADB=\angle CDB$ [I 4] Now, of two $\triangle$s ADO and CDO, the two sides AD, DO are equal to CD, DO respectively and $\angle ADO=\angle CDO$, $\therefore \angle AOD=\angle COD$ [I 4] &c

3 Let the diagonals AC, BD cut each other at O

$\triangle ABC$ is equal to $\triangle BCD$ in every respect

$\therefore \angle BAC=\angle CDB$ [I 4]

$\triangle BCD=\triangle ACD$, $\therefore \angle CDB=\angle ACD$, [I 4]

But $\angle BAC=\angle BCA$ [I 5]

$\therefore \angle BCA=\angle ACD$ $\therefore BO=DO$, and $\angle BOC=\angle DOC$, [I 4.]

$\therefore OC$ is at right angles to BD (*Def* 11) Similarly $AO=OC$, &c.

Prop 6

The angle BCF is equal to the angle CBG, therefore BH is equal to CH [I 6], &c Apply Euc I 4

Prop 7

Join the vertices of the two triangles and apply Euc I 5.

Prop 8

1 Let ABC and ABD be two isosceles triangles on the opposite sides of AB Join CD Of two triangles ACD and BCD, the two sides AC, CD are respectively equal to BC, CD and $AD = BD$, ∴ $\angle ACD = \angle BCD$ and $\angle ADC = \angle BDC$ [I 8]

2 Join the centres with the points of intersection of the two circles, and then apply Ex 1

Prop 9

1 The line, joining the vertices bisects the vertical angles [I 8 and 4] Therefore if one of the triangles formed by the line joining the vertices be turned round that line, it will coincide with the other

2 Because AF bisects the angle A, ∴ it bisects DE at right angles (Ex 5 Prop 4) ∴ AF is the axis of symmetry of D and E

3 Bisect the vertical angle

4 This is evident from Prop 4

5 Let the $\angle ABC$ be bisected by BD and the $\angle ACB$ by CD. $\angle ABC = \angle ACB$, (I 5),

$\angle DBC = \angle DCB$, ∴ $BD = DC$

Now apply Ex 1

Prop 10

1 Bisect AD at E and DB at F, AB is divided into 4 equal parts, &c &c

2 With A as centre and AB as radius describe the ⊙ BCD, with B as centre and BA as radius describe the circle ACE cutting the former circle at C and F Join CF cutting AB at G AB is bisected at G

3 Take any point E such that $AE=BE$ Join ED ED bisects AB at right angles [I 8] $\angle EDB$ is a right angle But $\angle CDB$ is a right angle ∵ ED coincides with CD, ∴ E is in CD

4 Apply Euc. I 8

Prop. 11

1 For all right angles are equal

2 Draw a straight line through the middle point of the line joining the two given points at right angles to the same and let it meet the given line Join the point of intersection with the two given points The problem is impossible when the straight line joining the points is at right angles to, and is not bisected by, the given straight line

3 Let ABC be any triangle Let the straight line bisecting at right angles the sides AB, BC meet at D The point D is equidistant from A, B, C

4 Let C, D be two points on opposite sides of AB Let CE be perpendicular to AB and produce CE making EF equal to EC Join DF, produce DF to meet AB at G Join GC The solution is not possible when EF is equal to the distance of AB from D

5 Let AB be the given line Draw any line AC Bisect the angle BAC by AE From AC cut off $AD = AB$, draw DF at right angles to AC meeting AE at F Join FB Apply Euc I 4

6 If the triangle DFC be turned round FC, the line DF will coincide with FE, ∴ perpendicular to one of them is also perpendicular to the other

Prop. 12.

1 Apply Euc I 4 2 Apply Euc I 4

8 Apply Euc I 8

Prop 13

2 The adjacent interior angles are equal, ∴ the sides are equal (I 6)

Prop 14

1 Let E be the middle point of the diagonal AC of the rhombus $ABCD$ By Euc I 8 and Def 11, BE is at right angles to AC, likewise DE is at right angles to AC &c

2 Apply Euc I 13 and 14

Prop 15

1 If four lines AE, DE, BE, CE, meeting at a point E make the vertical angle AED = the vertical angle BEC and $\angle AEC = \angle BED$, then AE is in the same straight line with EB, and DE is in the same straight line with EC The angles AED, AEC are together = angles BEC, DEB = two right angles (Cor 2), &c &c

2 Apply Euc I 13, 14.

Prop 16

1 Apply Euc I 5 2. Apply Ax 11

Prop 17

1 Take any point E in BC, join AE The angle AEC is greater than the angle B, and the angle AEB is greater than the angle C ∴ the angles AEC, AEB or two right angles are greater than the angles at B and C, &c

2, Let ABC be any triangle The angles at A, C are less than two right angles, the angles at B, C are together less than two right angles, etc

3 Apply Euc I 13

Prop 18

1 Apply Euc I 5 and 16

2 (1) when D is within CB

The angle ADB is greater than the angle at C, and the angle ADB is equal to the angle ABD

(2) when D is in CB produced

The angle ABD= angle ADC, but ABC is greater than ADC ∴ the ∠ ABC is greater than the angle ABD But ABD is greater than the angle ACB ∴ ABC is greater than ACB

3 Let BAC be a triangle, BC being not less than AB or AC

The angle BAC is not less than the angle ABC Therefore the angle ABC is acute [I 17] Likewise the angle ACB is acute

4 Draw the diagonals

Prop 19

1 Bisect BC at D, produce AD to K making $DK=AD$

By Euc I 4 $BK=AC$ and the ∠DBK=∠ACB

The angle ABC is greater than the angle DBK Therefore BE which bisects the angle ABK falls above BD Let BE meet AD at E AE is less than AD, ∴ AE is less than EK From EK cut off $EG=AE$

Produce BE to F making $EF=BE$ Join FG Produce FG to meet BK at H By I 4, $AB=FG$ But the angle BFH=angle ABE=angle FBH, ∴ $BH=FH$ ∴ BH is greater than FG ∴ BH is greater than AB But $AC=BK$ and is greater than BH. ∴ AC is greater than AB

2 The angle DBC is greater than the angle DCB, &c.

3. Additional Prop. VI

Prop 20

1. Apply Euc I 5, 16 and 19

4. Produce the line making the produced part equal to the line; join its extreme point with any opposite angle; and apply Euc. I 20

5 Each median to a side is less than half the sum of the other two sides, &c

6 Let AB be the given straight line and C, D the two points From C draw CE perpendicular to AB, and produce CE to F making EF equal to CE. Join FD cutting AB at H H is the required point

Prop 21.

1. Apply Euc 1 16.

2 Every two of these three lines drawn from the extremities of each side are less than the other two sides (I. 21) Add these and take their halves.

Two of these three lines are greater than the side with which they make a triangle (I. 20) Add these and take their halves

Prop 22

2. Divide the given figure into triangles

Prop 23

1 Cut off the arms of the angle making them equal to the two given sides

2 At both extremities of the base and on the same side of it make angles equal to the given angles

3 Let AB be the given base and ABC the given angle Make BC equal to the sum of the sides Join AC At the point A and in the straight line CA make the angle CAD equal to the angle ACB ADB is the required triangle

4 Let AB be the base and BAC the given angle Make AC equal to the given difference Join BC and produce AC to D. Make the angle CBD equal to the angle BCD.

Prop 24

At the point A in the straight line AB make the angle BAK =the angle BAH on the other side of AB. Make $AK=AH$ or AC, and join BK The angle BAC is greater than the angle BAK, therefore the bisector of the angle KAC will fall to the right of AB Let AQ the bisector meet BC at Q Join KQ The angle ACQ is equal to the angle AKQ and the angle AKB=the angle AHB [I 4] But $\angle AKQ$ is greater than $\angle AKB$

$\therefore$ $\angle AKQ$ or $\angle ACB$ is greater than $\angle AKB$ or $\angle AHB$.

Prop 25

On AC describe the triangle ACG, so that AC, CG, AG be respectively equal to DF, EF, DE (I 22) BC is greater than CG or EF, $\angle BGC$ is greater than $\angle GBC$. Make $\angle BGH = \angle GBC$, GH cutting BC at H By Euc I. 8, the angle BAH = the angle GAH. $\angle BAC$ is greater than the angle GAH, &c.

OTHERWISE

From BC cut off $BG = EF$ On BG describe the $\triangle BHG$, on the other side of BC, so that BH be equal to DE, and $HG = DF$ Produce HG to meet AC at K HK is greater than AK, &c &c

Prop 26

1 Let the hypotenuse AC and the side AB of the right-angled triangle ABC be equal to the hypotenuse DF and the side DE of the right-angled triangle DEF If the triangle DEF be applied to the triangle ABC so that D falls on A and DE on AB, then DE will coincide with AB and the point E with B, for $DE = AB$

$\angle ABC = \angle DEF$ (Ax 11), EF will fall on BC EF is equal to BC, if not, cut off $BG = EF$, &c &c

2 Let one triangle be applied to the other, so that the hypotenuse of one falls on the hypotenuse of the other, the sides forming equal acute angles with the hypotenuse will coincide &c &c.

3 Apply 1st case 4 Apply 2nd case

5 From the given point drop a perpendicular to the line bisecting the angle between the two given straight lines and produce it both ways, if necessary, to meet the lines, apply Euc I 26

6 Let A, B, C be the three points Join BC and bisect it at O Join AO and produce it to D AD is the required line

Prop 27

1 Apply Euc. I 8 3 Proceed as in Ex 1

Prop 29.

4 The parts of any two perpendiculars and the parts of the parallel lines intercepted form a rectangle Apply Euc I 26

5 Apply Euc. I 26

Prop. 31

1 Let A be the given point and BC be the given line At any point B in BC make the angle CBD equal to the given angle. Through A draw AE parallel to DB

2. Let D be the middle point of BC, the base of the triangle ABC. Let another triangle be formed with the same vertical angle A, by drawing EF through D meeting AB, AC, produced, if necessary, at E and F respectively. Through B draw BG parallel to AC to meet FE at G. Apply Euc I 26.

3 Proceed as in Ex 4 page 111.

4 Let AB, CD be the two parallel straight lines and let E be the given point. Take any point A in any of the lines. From the centre A at the distance equal to the third given line describe a circle cutting CD at F. Through E draw a straight line parallel to AF, etc.

The construction fails when the given line is less than the perpendicular distance between the parallel lines.

5 Apply Euc I 29 and 6.

6 From D a point in BC the base of the isosceles $\triangle ABC$ DEF is drawn at right angles to BC cutting AB at F and CA produced at E. AEF is an isosceles triangle. Draw AG parallel to FD meeting BC at G. Apply Euc I 29, 26, 6.

7 Let AD, CB be two parallel straight lines and A, B, two points. Join AB. Make the $\angle BAC = \angle ABC$ and the $\angle ABD = \angle BAD$.

8 Bisect one of the base angles of the isosceles triangle ABC by BD the bisector meeting opposite side at D. Draw DE parallel to BC, &c.

9 Bisect the base angles of the triangle ABC by BE and CE. Through E draw a straight line parallel to BC cutting AB at F and AC at G. $FBCG$ is the required figure.

Prop 32

2. See Additional Prop IV, page 91.

3 See Ex 2 page 110.

4 The perpendicular from the vertex bisects the vertical angle, &c.

5 Let ABC be the right angled triangle and the angle at C double the angle at A. Produce BC to E so that $BE = CB$. $\therefore \triangle ABC = \triangle ABE$ (I 4) $\therefore \angle C = \angle E$, and $\angle BAC = \angle BAE$, or $\angle CAE = 2\angle BAC = \angle C = \angle E$ $\therefore \triangle ACE$ is equilateral.

6 Apply Euc I 28.

9 Apply Euc I 18.

10 Apply Euc I 32, cor 1 and 2.

Prop 33

1 See Additional Prop II, page 90.

2 Let $ABCD$ be the quadrilateral, and E, F, G, H, the middle points of AB, BC, CD, DA, respectively ∴ by the preceding Ex, $EH \parallel BD$, and $BD \parallel FG$, ∴ $EH \parallel FG$ Similarly $EF \parallel AC \parallel HG$.

Prop 34.

1 Apply Euc I 29 and 26

2 Apply Euc I 4 and 27.

5 See Additional Prop III, page 90

6 See Additional Prop IV, page 91

7 Let AB, CD be equal and parallel, let EF be any other straight line Draw AG, BH, CK, DL perpendiculars to EF. Draw $AM \parallel EF$ meeting BH at M and $CN \parallel EF$ meeting DL at N, &c

8 Proceed as in Ex 7

10 See Ex 4 of Prop 31

11 See Ex 4 page 111

12 From the given point draw a line to the middle point of one of the diagonals, and produce it to meet the opposite side or sides

Prop 35

2 Let ABC be a triangle Let $BAFD$ and $ACFG$ be parallelograms on AB and AC respectively DE meets FG at H The side BK of the parallelogram $BKLC$ adjacent to BC is equal and parallel to HA

Produce HA to meet KL at M

▭DA = ▭BH = ▭BM, &c

3 Through the middle point of any of the other sides of the trapezoid draw a straight line parallel to the opposite side

Prop 37.

1 Let AB be the side of the trapezium $ABCD$ which should also be a side of the triangle required Through C draw CE parallel to BD, produce AD to meet CE at E ABE is the required triangle

2 Through the vertex of the given triangle draw a straight line parallel to the base meeting the given straight line

3 Their altitudes are equal

4 The line drawn through the vertex parallel to the base is the required locus

Prop 38.

2 Divide the base into parts equal to the number of equal parts into which the triangle is to be divided Join the vertex with the points of division.

3 This may be proved indirectly

8 See Ex 5, page 112

9 Join GC cutting AB at the point O GO is equal to OC, &c , &c

10 Apply Additional Prop II

11 This may be proved indirectly

12 Trisect the base and proceed as in Ex 8

Prop 39.

3 Add the triangle BEC to the equal triangles, then the triangles ABC, DBC are equal

Prop 40

4 The perpendiculars from the vertices on the line containing the bases are equal

5 Apply Additional Prop II

7 Through one of the middle points draw a line parallel to the other non parallel side

Prop 41

1 Let $ABCD$ be a square Produce CB to E making BE equal to BC Join AE AC

3 Through the point draw a straight line parallel to the bases

Prop 42

1 Produce the side CD of the parallelogram $ABCD$ to E making DE equal to CD at the point C in the straight line EC make the angle ECF equal to the given angle meeting AB at F. Join FE

2 Bisect the side AC at D Through the opposite angular point B draw BE parallel to AC From D as centre and with radius equal to half the sum of AB, BC, describe a circle cutting BE at E AE is the required parallelogram

3 Trisect AB at E, F draw EG, FH parallel to AD or BC meeting DC at G and H Let the given point P be in EF The lines drawn through P through the mid points of EG, FH will trisect the parallelogram If the given point P be in FB, the

straight line through the middle point of FH will cut off one-third of the parallelogram and the remaining trapezium $APMD$ is to be bisected by a line from P. Draw MK parallel to DP meeting AD produced at K, bisect AK at R and join PR.

Prop 43

1 Let EH, BD cut AC at O, P. Join OB, OD. Apply Euc. I. 34, Ex 1, 38 and 39.

2 The diagonals divide the angles into half right angles.

3 This is to be proved indirectly.

Prop 44

1 Let ABC be the given triangle. Draw BD at right angles to BC making BD=given altitude. Draw DE parallel to BC meeting BA or BA produced at E. Draw AF parallel to EC meeting BC produced or BC at F. EBF is the required triangle.

2 Describe a triangle equal to double the given parallelogram, and if AB be the given straight line, to AB apply the parallelogram BI (Euc I 44) equal to the triangle. Make the angle BAM equal to the given angle. ABM is the required triangle.

Prop 45

1 FM is a rectangle when L is a right angle.

2 Let $ABCD$ be the parallelogram. Bisect BD at E. Through E draw GEF at right angles to BD.

Through A and C draw AG and CF parallel to BD. $BFDG$ is the required rhombus.

3 Construct a parallelogram on the base equal to the given triangle and having an angle equal to a right angle.

Prop 46

2 See Prop 1 page 82.

3 The square described on double the side of the given square is the required one.

Prop 47.

1 Let ABC be a right-angled triangle, the acute angle C being double of the acute angle A. Let BD be a median to AC. BCD is an equilateral triangle. The square on AC is four times the square on CD, &c.

3 Bisect AB the given line at C. Bisect BC at D, on BD describe a square. With C as centre and CB or CA as radius describe a circle; with B as centre and the diagonal of the square described as radius, describe a circle cutting the other circle at E. AEB is the required triangle.

4 Apply Euc I 4.

5 Draw DE perpendicular to BC from D the middle point of AB

$$EC^2-BE^2=EC^2+DE^2-DE^2-BE^2$$
$$=DC^2-BD^2=DC^2-DA^2=AC^2.$$

6 Each of the angles FAB, $KAC=\frac{1}{2}$ a right angle, &c

7 Place the sides of the two given squares at right angles to each other and join their extremities

8 Take a line AB equal to the side of the smaller square Draw any line BC at right angles to AB From the centre A and with radius equal to the side of the larger square describe a circle cutting BC at D

9 Let AB be the given straight line At the point A in BA make the angle BAE equal to half a right angle From B as centre and with radius equal to the side of the given square, describe a circle cutting AE at L Draw LD perpendicular to AB AD, DB are the two parts

10 Let AB be the given straight line Draw AC at right angles to AB At B make the angle $ABC=\frac{1}{2}$ of a right angle At C make the angle $DCB=$ CBA cutting AB at D

11 The diagonals bisect each other at right angles

12 Produce DB, FC to meet AG, KH at M and N The triangle FBM may be proved equal to ABC and to FBD (Euc I 26 38)

13 Take any point P in CP which is perpendicular to AB The difference between the squares on PA, PB is equal to the difference between the squares on AC CB

14 Let D be the middle point of AC the hypotenuse of the right angled triangle ABC

The squares on the sides $=2AC^2=8AD^2=8BD^2$

15 Describe equilateral triangles ABD, BCE, CAF on the sides AB BC CA respectively of the right angled triangle ABC, having the angle at A a right angle Let angle $ACB=\frac{2}{3}$ of a right angle Draw AH, EK perpendiculars to BC DBC is a right angle The $\triangle ABD=\triangle DBH$, $DB=EK$, $\triangle DBH=\triangle BHE$ &c, &c

The restriction that one of the acute angles is equal to two-thirds of a right angle may be removed and the exercise may be solved in the following manner —

Draw DG perpendicular to AB and AH, EK perpendiculars to BC Join DC, AE, AK, EH, GC $\triangle DBC=\triangle ABE$

$$\triangle DBG=\triangle AGD=\triangle DGC,\ \triangle AKF=\triangle HKE,$$

$\triangle ABK=\triangle BGC$

$\triangle DBC = \triangle DBG + \triangle DGC + \triangle BGC = \triangle ABE = \triangle ABK + \triangle BKE + \triangle AKE$

$\therefore \triangle ABD = \triangle DBG + \triangle DGC = \triangle BKF + \triangle AKE$
$= \triangle BKL + \triangle EKH = \triangle BHE$ &c, &c.

Prop 48

1 & 2 Apply Euc I 25

3 $BC^2 = 4 AB^2 = AB^2 + AC^2$

Prop XIII

2 The diagonal (with its produced part) through the vertex of the angle at which the adjacent sides meet is the required locus

3 The line through the vertex parallel to the base is the locus to be found out

Prop XIV.

1 Let the line bisecting at right angles the line CD joining the given point C, D meet the given line AB at F F is the required point

2 See Ex 1, page 109

3 Let A be the given point and BC the given straight line On the same side of BC in which A is, draw $DF \parallel BC$ at the given distance With A as centre and the given distance as radius describe a ⊙ cutting DF at G G is the required point

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Miscellaneous Exercises on Book I

1 DB is equal to AE because each of them is double of AB, and AC is equal to BC, etc Apply Euc I 4

2 From any of the two given points A, B draw AC perpendicular to the given straight line, produce AC and make CD equal to AC Join any extremity of this line with B so as to cut the given straight line The point of intersection is the required point

3 Euc I 13 and Ax 3 4 Euc I 6

5 Euc I 13 and 6 6 Apply Euc I 4 and 8

7 Join the points so as to form a triangle From the middle points of any two sides of the triangle draw straight lines at right angles to them intersecting at a point This point is the centre of the required circle

8 Bisect BC at E and draw ED at right angles to BC meeting BA or BA produced at D

9 From the given point A draw AD perpendicular to the given straight line BC. From the centre D and with any radius, describe a circle cutting BC, produced if necessary, at the points E, F. Join AE, AF. AE is equal to AF.

10 Euc. I. 14.

11 Let AB be a side and ABC the given angle. With A as centre and the other given side as radius describe a circle cutting BC (when AB is the smaller side) at D, or (when AB is the longer side) at D and D'.

12 Let AB be the given side, BAC the given angle, and AC the given difference. Produce AC to E, and join BC. At the point B in BC make the angle CBE equal to the angle BCE (I. 23). AEB is the triangle required.

13 Apply Euc. I. 5 and 8.

14 The two given points must be on the opposite sides of the given straight line and they must not be equidistant from the given line. Proceed as in Ex. 2.

15 From any point A in any line BC draw AD at right angles to BC, and make AD equal to the given perpendicular. From D as centre and with radius equal to the given side describe a circle cutting BC at C. Draw DB at right angles to DC.

16 Apply Euc. I. 19 and 6. 17 Euc. I. 20.

18 Apply Euc. I. 4 for the solution of the first and I. 8 for the second case.

19 Apply Euc. I. 26.

20 From the given point draw a line to the middle point of the line joining the other two points, drop perpendiculars on this line and apply Euc. I. 26.

21 From A the first given point draw AD to D the middle point of the line joining the other two points B, C. From B, C drop perpendiculars BE, CF to EF drawn through A at right angles to AD. Through B, C draw parallels to EF and apply Euc. I. 26 and 34.

22 Draw AC at right angles to the given side AB; make AC equal to the given sum or difference. At the point B in the straight line CB make the angle CBD equal to the angle at C meeting CA or CA produced at D (I. 23).

23 Apply Euc. I. 26.

24 Let D be the middle point of BC the base of the triangle ABC; E, F are points in BC at equal distances from D. Produce AD to G making DG equal to AD. Join BG, EG. Apply Euc. I. 21.

25 Apply Euc. I. 4 and 26.

26 From the given point A draw any line AB making it equal to double the least. On AB describe the triangle ABC so that AC

and BC may be equal to the other two given lines (I 22) Through A, B draw AD, BD, parallels to BC AC respectively Join CD cutting AB at E AC, AD and AE are the required lines

27 Let AB be the given side, D its middle point, draw DE at right angles to AB, make DE equal to the given perpendicular. At the point A in the straight line BA make the angle BAC equal to the given angle From E draw EF perpendicular to AC, make FC equal to AF Join BC BCA is the required triangle

28 From any point C in a straight line AB draw CD at right angles to AB, make DC equal to the given perpendicular From CA, CB cut off CE CF equal to the given differences Join ED, DF Make the angles EDH, FDG equal to the angles DEH, DFG respectively DGH is the required triangle

29 Let AB, AC be the given lines draw AD perpendicular to AC, making AD = given difference Draw DEF parallel to AC cutting AB at E The bisectors of the angles BEF BED make up the pair of lines which is the required locus

30 The diagonal joining the extremities of the adjacent sides of the rectangle is the required locus

31 The straight line at right angles to the given line from the middle point of the portion between the intercepts is the required locus

32 Apply Euc I 13 and 28

33 Bisect the angle ABC by BD cutting AC in D D is the required point Euc I 29 and 6

34 Apply Euc I 26 and 4   35 Euc I 4 and 27

36 D is the middle point of AB Apply Euc I 38

37 Let the two straight lines AB AC meet at A Join A with the given point D Produce AD to E making DE equal to AD Through E draw EC parallel to AB Join CD and produce it to meet AB at B CB is the required line Euc I 29 and 26

38 Apply Additional Prop 11

39 Through the middle points of non-parallel sides draw perpendiculars to the parallel sides

40 Apply Euc I 32

41 Let ABC be one of the angles and AB be the side opposite to the other angle At any point C in BC make the angle BCD= the other angle on the same side of BC on which AB is. Draw $AE \parallel DC$

42 Apply Euc I 32   43 Apply Euc I 32

44 The opposite sides are equal, by Euc I 4 Apply Euc I. 8 and 27.

45 Halves of the exterior angles at A, C are together equal to the angles at B and D (I 32), etc.

46 Apply Euc I 32

47 Let ABC be the right angled triangle, and the angle at C is treble the angle at A. At the point C in the straight line AC make the angle ACD equal to the angle at A. Bisect the angle DCB.

48 By Euc I. 32, each of the equal angles may be proved equal to the opposite vertical angle.

49 Let ABC be the isosceles triangle, A being the vertex. At the points B, C in the straight lines AB, AC respectively make the angles ABD, ACD each equal to half the angle at A. BCD is the triangle required. The problem is impossible when the vertical angle is a right angle or greater than a right angle.

50 Let AC be the given hypotenuse. In AC take AD equal to the given difference. Bisect CD in E and AC in F. From the centre F and radius equal to CF or AF, describe the circle ABC. From E draw EB at right angles to AC, meeting the circle at B. Join AB, BC. ABC is the required triangle.

51 Describe an equilateral triangle ABC. Draw AD perpendicular to BC. If AD be not equal to the given perpendicular, from AD or AD produced cut off AE equal to the given perpendicular. Through E draw a straight line parallel to BC meeting AB, AC, produced if necessary, at the points F, G. AFG is the required triangle.

52 Let AB be the given side; from B draw BC at right angles to AB making it equal to the given sum. Join AC. At the point A in the straight line AC make the angle CAD equal to the angle at C. ABD is the triangle required.

53 Let D be the required point in AC, so that the straight lines DE, DF drawn parallel to the sides BC, AB respectively form the rhombus $BEDF$. Then the angle FBD may be proved equal to DBE. Hence the construction.

54 Let ABC be the given triangle and DE the sum of the three sides of the required triangle. At the points D, E make the angles EDF, DEF equal to the angles at B and C respectively. Let DO, EO bisect the angles at D and E. Through O draw OH, OG parallels to DF, EF respectively. OHG is the required triangle.

55 The parallel lines shall cut off equilateral triangles.

56 Apply Euc I 5, 32 and 27.

57 Let ABC be the required triangle. From CB cut off CD equal to CA. Join AD. The angle at B is half a right angle, and the angle ADC is three-fourths of a right angle; hence the construction. Let BD be the given difference; produce it to C. At the points B, D make the angles DBA, CDA equal to half and three-fourths of a right angle respectively. Draw AC at right angles to AB meeting BC at the point C. ABC is the required triangle.

58 Suppose the square $DEFG$ to be inscribed in the equilateral triangle ABC, so that EF lies upon BC and the points D, G lie on AB, AC Draw AH perpendicular to BC Join AF The triangle ADG is equilateral, therefore AG is equal to FG, for it is equal to DG Therefore the angle GAF is equal to the angle GFA (I 5), which is equal to FAH (I 29) Therefore AF bisects the angle GAH Hence the construction Draw AH perpendicular to BC Bisect CAH by AF meeting BC at F Through F draw FG parallel to AH, through G draw GD parallel to BC, and DE parallel to AH or GF

59 This is the same as Ex 54, one of the angles of the given triangle being a right angle and the other two each half a right angle

60 Let AB be the given sum and ABC the given angle From A draw AC perpendicular to BC Bisect the angle at A by AD meeting BC at D Draw DL parallel to AC DLB is the required triangle

61 Let ABC be the required triangle, having the base BC equal to the given base, the angle at A equal to the given angle, and BD along BA equal to the difference of AB and AC Join DC The angle ADC is equal to the angle ACD Therefore the angle ADC is equal to half the supplement of the angle at A Hence the construction

62 Let ABC be the triangle Let the angles at B and C be acute Take any two points D, E in BC At the points D, E make the angles GDC, FEB each equal to half a right angle, DG meeting AC at G and EF meeting AB at F Bisect DG, EF at H, K respectively Join HC, BK intersecting each other, produced if required at O Through O draw LM, parallel to DG meeting BC at L and AC at M Through O draw NP parallel to EF meeting AB at N and BC at P Join LN, NM, MP, $NLPM$ is a square and it is inscribed in ABC By Ex 3, Prop 38, LO is equal to OM and NO to OP

63 Describe an equilateral triangle ABC Bisect any two of its angles B, C by BO and CO BOC is the required triangle

64 Let $ABCD$ be the given square Join BD At the point B in DB make the angle DBE equal to half the angle of an equilateral triangle, BE meeting AD at E, also make the angle DBF equal to the same angle, BF meeting DC at F, Join EF BEF is the required triangle

65 Let AB be equal to the sum of the sides At B in AB make the angle ABC equal to half the given vertical angle From the centre A at the distance equal to the given base describe a circle cutting BC at C At the point C in the straight line BC make the angle BCD equal to the angle at B, and let CD meet AB at D ADC is the required triangle

66 Apply Euc I 29, 6 and 32

67 Apply Euc I 32.

68 Through the given point draw a straight line at right angles to the bisector of the angle made by the straight lines

69 Draw a diagonal and apply Euc I 8 and 27

70 Apply Euc I 32, Cor 1 and 28

71 Let $ABCD$ be a rectangle It is a parallelogram (Euc. I 28), therefore the opposite sides are equal Apply Euc I 4

72 Through D draw DF parallel to AB meeting BC at F AB is equal to DF (I 34) Therefore DF is equal to DC The angle DFC is equal to DCF, &c

73 Through any point G in AC draw GH parallel to CD meeting BD produced at H CG is equal to DH

74 Let B be the line to which the required line should be parallel and D to which it should be equal Let the other two straight lines FA, AG meet at A Draw AE parallel to B and make it equal to D Draw EF parallel to AG meeting AF at F Draw FG parallel to AE meeting AG at G FG is the required line

75 The point where the line bisecting the vertical angle meets the base is the required point

76 Let E be the given point in AB a side of the parallelogram $ABCD$ Join E with O the middle point of AC and produce EO to meet DC at F Through O draw GOH at right angles to EO meeting BC at H and AD at G $EGFH$ is the inscribed rhombus

77 See Ex 26 78 Apply Euc I 28 and 34

79 Through the points of bisection draw parallels to the diagonal and apply 33, 26 and 34

80 This is the same as Ex 75

81 The sum of each pair may be shown to be double of the perpendicular from the point of intersection of the diagonals, by drawing through the point a straight line parallel to the given straight line so as to meet the perpendiculars

82 Let $ABCD$ be the parallelogram Of the two adjacent sides let BC be not less than its distance from AD From B as centre and BC as radius describe a circle cutting AD at F Join BF, draw CE parallel to BF meeting AD at E E is the required point

83 By applying Euc I 4, the opposite sides may be proved equal Then apply Ex 69

84 From any point in the middle line draw straight lines parallel to the others so as to form a parallelogram The other diagonal is one of the required lines

85 This is proved in Ex 81

86 At the distance of the altitude draw a straight line parallel to the base From the middle point of the base as centre and

radius equal to the line bisecting the base, describe a circle cutting the parallel line The point of intersection is the vertex of the triangle

87 The triangles may be placed between the same parallels.

88 Let D be the middle point of AB Apply Euc. I 37 to show that BEL is equal to CDB, etc

89 Prove this indirectly by applying Euc I 38

90 Apply Additional Prop II, page 90

91 Join DC draw BE parallel to DC Apply Euc 37

92. Apply Euc I 37

93 Let $ABCD$ be a parallelogram and E a point without it. Through E draw EFG parallel to AB or CD cutting AD at F and BC at G Join BF, FC LCD is equal to FCD, which is equal to FBD (Euc I 37), also AEF is equal EBF

94 Join the points Through each point draw a straight line parallel to the opposite side

95 See Exercise 92

96 The straight line drawn through one of the vertices parallel to the base is the required locus

97 From the extremities of the base AB as centres and radii equal to two thirds of the given bisectors, describe two circles intersecting at O Produce AO to E, making OE equal to half of AO also produce BO to F making OF equal to half of BO Join AF, BE and produce them to meet at C See Additional Prop XVII. page 102

98 Prove by superposition

99 Produce BA to K making AK equal to BA Draw CH parallel to PK Join PH cutting CD at F and AD at F The triangle ADC is equal to the triangle AKC, and HKC is equal to HCP (I. 37)

100 AC is trisected in F (See Ex 79) Apply Euc I 38

101 Join BD cutting PQ at O The triangle BOP is equal to the triangle QOD and the triangle BQP is equal to the triangle QPD

102 Join any two of the angular points with the middle points of the opposite sides The point of intersection is the required point

103 Through C the vertex of the triangle ABC draw CD parallel to AB Bisect AB at E, and draw EF at right angles to AB meeting CD at F Join AF, BF

104 Let GE, FC intersect each other at B Draw BD at right angles to GE, make BD=given straight line, draw $DC \parallel GE$ meeting BC at C From BE cut off $BH=BC$, from BG cut off

$BG=BO$ and from BF cut off $BF=BO$ CH, FH, FG, GC make up the required locus.

105 Apply Additional Prop III, page 90

106. The square on the diagonal is the required one.

107 A line drawn perpendicular to the given base is the required locus

108. Apply Euc. I 47

109 Let ABC be a right angled triangle, the angle at B being a right angle; let D be the middle point of AC, let the square on AB be three times the square on BC AC may be proved double of BD or BC, and DBC an equilateral triangle.

110 Produce FB, KC Apply Euc I 32

111. Each of the alternate angles is equal to half a right angle.

112 Apply Euc I 26

113 Draw DM, EN perpendiculars respectively to FB, KC produced Apply Euc I 47

114 Produce DB to meet FG at R, ML may be proved parallel to BD

115. Let ABC be the triangle Let perpendiculars AD and BE meet at O The perpendicular CF from C shall pass through O If it do not pass let it pass otherwise Draw OH perpendicular to AB Join OC The difference of the squares on AC, CB is equal to the difference of the sum of the squares on AC AB and of BC, AB Prove that this is equal to the difference of the squares on AO, OB Hence prove that the points F, H must coincide

116 Let ABC be the right angled triangle, AB being the hypotenuse and let the angle ABC be double of the angle CAB On AB describe the equilateral triangle ADB and on BC the equilateral triangle BEC Bisect DB at G Join AG, GC The triangle ADG is equal to the triangle AGB, which again may be proved equal to the triangle ABC The triangle AGC is equilateral and it is on AC GB is in the same straight line with BE The triangle GBC is equal to the triangle BCE The figure $AGBC$ is equal to the equilateral triangles on AC, BC and also equal to the triangle ABD

117 Apply the proof of Euc I 47

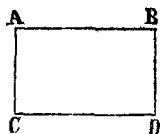
118 The side AB is produced to D and AC to E The angle CBD is bisected by BO and BCE by CO OD, OP, OE perpendiculars on AD, BC, CE respectively are equal (I 26) AD may be proved equal to AE (I. 47). Euc I 8

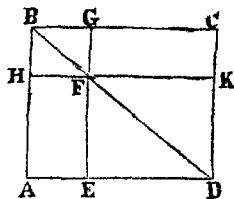
BOOK II.

DEFINITIONS

1 A **Rectangle** is a parallelogram whose angles are right angles.

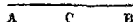
It might be seen that if one angle of a parallelogram is a right angle, all the angles must be right angles. [Euc. I. 29.

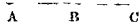
To determine a rectangle it is only necessary to know two sides which are conterminous, for the other sides being opposite to these are equal to them It is usual, therefore, to express a rectangle by its two conterminous sides Thus, if AB, AC express two sides which are conterminous, the rectangle is called "the rectangle under AB, AC," or more briefly, "the rectangle AB, AC." Of course this is only a short way of saying "the rectangle contained by the sides AB, AC"

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a **gnomon**

Thus the parallelogram HG together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make up the gnomon.

The Second Book of Euclid treats of the properties of rectangles contained by straight lines, divided into segments, or contained by the sides and segments of sides of any triangle.

When the sides of a rectangle are equal, the rectangle becomes a square.

When a straight line is divided into two parts, each part is called a segment. A straight line is generally divided into segments both *internally* as well as *externally*. When a point is taken in a straight line, the distance of this point from each extremity of the line is a segment, and the straight line is said to be divided *internally*. Thus the straight line AB is divided *internally* when any point C is taken in it and then AC and CB are the two internal

A C B

segments. When a straight line is produced, and any point is taken in the produced part, the distance of this point from each extremity of the line is a segment and the straight line is divided *externally*. Thus if a point C be taken in AB produced, AC and CB are the two external segments.

A B C

The properties of rectangles treated of in this Book of Euclid refer only to areas. It is important to notice that areas in Geometry are not denoted by such Arithmetical units as *square feet, square yards, square miles*, etc., nor are lengths denoted by *feet, yards, miles*, etc. Lengths are measured in terms of some given straight line or lines, and areas in terms of some rectangle contained by given straight lines. Thus in speaking of the area of the rectangle AB, BC, we do not know how many square yards there are in it, nor how many yards in the straight lines AB, BC; all that we are concerned with is the space occupied by the rectangle whose sides are the given straight lines AB, BC, *no matter how long or how short these lines may be*.

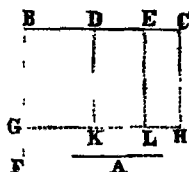
In the Second Book of Euclid an obvious axiom is made use of, viz. that a figure is equal to the sum of the component parts which make up that figure. Thus in the last page the figure ABCD is composed of the gnomon AGK and the figure DEFK, therefore ABCD is equal to the sum of these two figures.

For the future wherever possible we shall write everything in abbreviated language, since this is the more practical method. Thus in the last paragraph we have said that the figure ABCD is equal to the sum of the gnomon AGK and the figure DEFK; but we might have said in abbreviated language that ABCD = AGK + DEFK. But it must be remembered that this is only a *verbal abbreviation*, and has no connection whatever with Arithmetical or Algebraical formulas.

Proposition 1. Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

Let A and BC be two straight lines, and let BC be divided into any number of parts at the points D, E :

then the rectangle contained by the straight lines A, BC, *shall be equal to the rectangle contained by* A, BD, *together with that contained by* A, DE, *and that contained by* A, EC.

From the point B draw BF at rt angles to BC ; [I. 11.
and make BG equal to A. [I. 3.

Through G, draw GH par[l] to BC ;
and through D, E, C, draw DK, EL, CH, par[l] to BG. [I. 31.

The fig. BH is made up of the figs BK, DL, EH.

But the fig BH is the rect A, BC ;
since it is contained by GB, BC , and GB = A . [*Constr.*

and the fig. BK is the rect. A, BD ,
since it is contained by GB, BD , and GB = A · [*Constr.*

also the fig DL is the rect A, DE ;
since it is contained by DK, DE , and DK = A : [*Constr.*

likewise the fig. EH is the rect A, EC ;
since it is contained by EL, EC , and EL = A. [*Constr.*

Therefore, the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, by A, DE, and by A, EC.

Wherefore, *if there be two straight lines*, &c, Q.E.D.

EXERCISES.

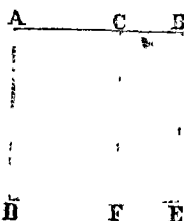
1 Given two straight lines, twice the rectangle under either of them and half the other is equal to three times the rectangle under either of them and a third of the other.

2 If each of two straight lines be divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the several parts of the one and each of the several parts of the other.

Proposition 2 Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of the parts are together equal to the square on the whole line

Let the straight line AB be divided into any two parts at the point C

Then the rectangle contained by AB, BC, *together with that contained by* AB AC, *shall be equal to the square on* AB

On AB describe the sq ADEB , [I 46.
and through C draw CF par[l] to AD or BE [I 31.

Then the fig AE is made up of the figs AF, CE

But the fig AE is the sq on AB ,

and the fig AF = rect. AB, AC

since it is contained by AD AC , and AD = AB

Also the fig. CE = rect AB, BC

since it is contained by BE, BC , and BE = AB

Therefore the rectangle contained by AB, AC, together with the rectangle contained by AB, BC, is equal to the square on AB

Wherefore, *if a straight line &c* Q E D

Exercises

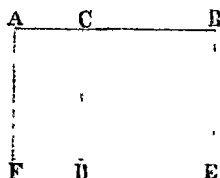
1 Prove Prop 2 without making any geometrical construction

2 ABC is a right angled triangle right-angled at A ; AD is drawn perpendicular to the hypotenuse BC , prove that the rectangles contained by BC, BD, and BC, DC, are together equal to the squares on BD, DC, together with twice the square on AD.

Proposition 3. Theorem.

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts

Let the straight line AB be divided into any two parts at the point C

then the rectangle contained by AB, BC *shall be equal to the square on* BC, *together with the rectangle* AC, CB.

On BC describe the square CDEB ; [I. 46.
and through A draw AF parl to CD or BE, meeting ED produced at F [I 31.

Then the fig AE is made up of the figs CE, AD

But the fig AE is the rect AB, BC,
since it is contained by AB, BE, and BE = BC · [*Constr.*
and the fig CE is the sq on BC
also the fig AD is the rect. AC, CB,
since it is contained by AC, CD, and CD = CB.

Therefore the rectangle AB, BC, is equal to the square on BC, together with the rectangle contained by AC, CB

Wherefore, *if a straight line* &c Q E D

EXERCISES

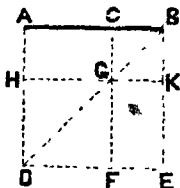
1 Prove Prop 3 without making any geometrical construction.

2 To produce a given straight line so that the rectangle contained by the whole line produced and the part produced may be equal to twice the square on the given line.

Proposition 4. Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts at C.

Then the square on AB *shall be equal to the sum of the squares on* AC *and* CB, *together with twice the rectangle contained by* AC, CB.

On AB describe the square ADEB, [I. 46.
join BD,
through C draw CGF par[l] to AD, or BE;
and through G draw HGK par[l] to AB or DE, meeting AD in H, and BE in K [I. 31.

Since the straight line BD meets the two par[ls] AD, CF,
∴ the ext. angle CGB = the int. and opp. angle ADB. [I. 29.

But the angle ADB = the angle ABD, [I. 5.
since AB = AD, being the sides of a square.

∴ the angle CGB = the angle ABD or CBG
CG = CB [I. 6.

And CB = GK, CG = BK. [I. 34.

Also, the angle CBK = a right angle. [I *Def.* 31.

∴ CK is a square, and it is described on BC

Likewise HF is the square on HG, that is, the square on AC,
since AC = HG. [I. 34.

Moreover, the complement AG = the complement GE. [I. 43.

And the fig. AG = the rect. AC, CB ; since CG = CB.

∴ the two figures AG, GE = twice the rect. AC, CB.

Now the square on AB is the figure AE.

And AE = the sum of its component figures

= the figs. CK, HF, AG, GE.

= the sum of the squares on AC, CB, together with twice the rect. AC, CB.

Therefore the square on AB is equal to the sum of the squares on AC, CB, together with twice the rectangle AC, CB.

Wherefore, *if a straight line*, &c Q. E. D.

COR From the demonstration it is manifest, that the parallelograms about the diameter of a square are likewise squares.

NOTE Alternative Proofs of Props. 4, 5, 6, 7, 8, 9, 10, 12 and 13 are given at the end of this Book

EXERCISES

1 Prove Prop. 4 without making any geometrical construction.

2. The square on any straight line is equal to four times the square on half the line

3 The difference of the squares on two unequal lines is equal to the rectangle contained by their sum and difference

4 If a straight line be divided into any number of parts, the square on the whole line is equal to the sum of the squares on all the parts, together with twice the rectangles formed by taking the parts two and two.

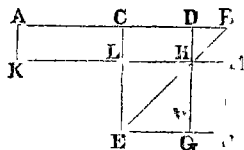
5. In a right-angled triangle, the square on the perpendicular from the right angle on the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse

6. In a right-angled triangle, portions are cut off from each end of the hypotenuse equal to the adjacent sides the square on the middle segment is equal to twice the rectangle under the extreme segments

Proposition 5. Theorem.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D.

Then the rectangle contained by AD, DB, *together with the square on* CD, *shall be equal to the square on* CB

On CB describe the square CEFB [I. 46.
join BE ,
through D draw DHG par[l] to CE or BF ,
through H draw KLM par[l] to CB or EF ,
and through A draw AK par[l] to CL or BM [I 31.

Since the complement CH = the complement HF [I 43.
to each equal add the fig DM
∴ the whole figure CM = the whole figure DF
But CM = AL, [I 36.
since AC = CB
AL = DF
to each equal add the fig CH
∴ the whole fig AH = the gnomon CMG

But AH = the rect AD, DB, since DH = DB.

∴ the rect. AD, DB = the gnomon CMG
to each add LG, i e the sq on CD, [II 4, *Cor.*

∴ the rect AD, DB, together with the square on CD
= the gnomon CMG together with the fig LG
= the whole fig CF
= the square on CB.

Therefore the rectangle AD, DB together with the square on CD, is equal to the square on CB.

Wherefore, *if a straight line*, &c. Q. E. D.

COR. From this proposition it is manifest, that the difference of the squares on two unequal lines AC, CD, is equal to the rectangle contained by their sum AD, and their difference DB.

EXERCISES.

1. Prove Prop. 5 without making any geometrical construction.

2. The rectangle contained by any two lines, together with the square on half their difference is equal to the square on half their sum.

3. Divide a given straight line into two parts such that the rectangle contained by them shall be the maximum (greatest possible).

4. If a perpendicular be drawn from the vertex of a triangle to the base, the rectangle contained by the sum and difference of the sides is equal to the rectangle contained by the sum and difference of the segments.

5. If from the vertex of an isosceles triangle a straight line be drawn to any point in the base, the difference of the squares on a side and on the line so drawn, is equal to the rectangle under the segments of the base.

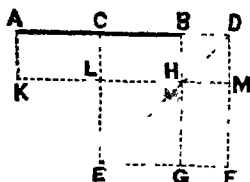
6. To divide a given straight line into two parts, so that the rectangle contained by the parts shall be equal to the square on another given straight line. What must be the condition that the solution may be possible?

7. The difference between the squares on the sides of a triangle is equal to twice the rectangle contained by the base and the part of it intercepted by the perpendicular on it from the opposite angle and the middle point of the base.

Proposition 6 Theorem.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected at C, and produced to the point D

Then the rectangle AD, DB, *together with the square on* CB, *shall be equal to the square on* CD.

On CD describe the square CEFD ; [I. 46.
join DE ;
through B draw BHG par[l] to CE or DF ;
through H draw KLM par[l] to AD or EF ,
and through A draw AK par[l] to CL or DM [I. 31.

The rectangle AL = the rectangle CH. [I. 36.

But the complement CH = the complement HF, [I. 43.

∴ the figure AL = the figure HF
to each equal add the fig. CM

∴ the whole AM = the gnomon CMG.

But AM = the rect. AD, DB, since DM = DB.

∴ the rect. AD, DB = the gnomon CMG .
to each add LG, *i. e.* the square on CB, [II. 4, *Cor.*

∴ the rect. AD, DB together with the square on CB
= the gnomon CMG together with the fig. LG
= the whole fig. CF
= the square on CD.

Therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD

Wherefore, *if a straight line &c.* Q.E.D.

EXERCISES

1 Prove Prop 6 without making any geometrical construction.

2 The difference of the squares on two lines is equal to the rectangle under their sum and difference

3 Produce a given straight line, so that the rectangle contained by the whole line produced and the part produced may be equal to three times the square on half the line

4 Produce a given straight line, so that the rectangle contained by the whole line produced and the produced part may be equal to the square on another given straight line.

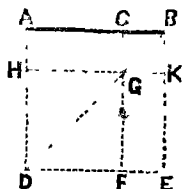
5 The difference of the squares on any two sides of a triangle is double of the rectangle contained by the third side and the projection of the median to that side on the same

6 If from the vertex of an isosceles triangle a straight line be drawn to any point in the base produced, the difference of the squares on a side and the line so drawn is equal to the rectangle under the segments of the base

Proposition 7. Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts, are equal to twice the rectangle contained by the whole line and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C;

then the squares on AB, BC, *shall be equal to twice the rectangle contained by* AB, BC, *together with the square on* AC.

On AB describe the square ADEB, and construct the figure as in Book II. Prop. 4.

Since the complement AG = the complement GE [I. 43.
to each equal add the fig. CK.
∴ the whole fig. AK = the whole fig. CE,
∴ the two figs. AK and CE = double of the fig. AK.
But AK = the rect. AB, BC, since BK = BC;
and the two figs. AK and CE = the gnomon AKF, together with the fig. CK or the square on BC;
∴ the gnomon AKF with the square on BC
= twice the rect. AB, BC:
to each equal add the fig. HF, *i. e.* the square on AC.
∴ twice the rect. AB, BC, with the square on AC
= the gnomon AKF, with the sq. on BC, and the fig. HF.
= the whole fig. AE, with the square on BC
= the square on AB, with the square on BC.

Therefore the squares on AB, BC are equal to twice the rectangle contained by AB, BC, together with the square on AC.

Wherefore, *if a straight line &c.* Q. E. D.

EXERCISES.

1. To divide a given straight line into two parts so that the sum of the squares on the whole line and on one part may be equal to twice the rectangle contained by the whole line and the other part.

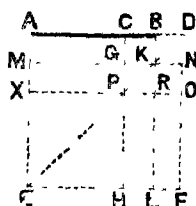
2 Prove Prop 7 without making any geometrical construction

Proposition 8. Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and that part

Let the straight line AB be divided into any two parts at the point C ,

then four times the rectangle contained by AB, BC, *together with the square on* AC, *shall be equal to the square on the straight line made up of* AB *and* BC *together*

Produce AB to D, so that BD may be equal to CB.

On AD describe the square AEFD , and construct two figures such as in the preceding Propositions

Then, because CB = BD = DN = NO ·
each of the figs CK, BN, KO, GR, may be proved to be equal, as being squares on the line CB or BD

Also, rect GL = rect. KF = rect AK = rect MR.
But the rect AK = the rect AB, BC .

∴ the sum of the four figs. GL, KF, AK, MR
= four times the rect. AB, BC.

But the fig. GL = the fig. PL, with the fig. GR, or BN.

∴ four times the rect. AB, BC

= the sum of the figs. PL, BN, KF, AK, MR,

= the gnomon AOH :

to each equal add the fig XH, *i. e* the sq. on AC,

∴ four times the rect AB, BC, with the sq. on AC

= the gnomon AOH together with the fig. XH

= the whole fig AF

= the square on AD

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is, to the square on the line made up of AB and BC together.

Wherefore, *if a straight line* &c Q. E. D.

EXERCISES.

1. Prove Prop 8 without making any geometrical construction.

2. If a straight line be divided into five equal parts, the square on the whole line is equal to the sum of the squares on the straight lines, which are made up respectively of four and of three of these parts

Proposition 9. Theorem.

If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together double of the sum of the squares on half the line and on the line between the points of section.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D;

then the squares on AD, DB *shall be together double of the squares on* AC, CD.

From C draw CE at right angles to AB, [I. 11.

and make CE equal to AC or CB. [I. 3.

join EA, EB;

through D draw DF par[l] to CE meeting EB at F, and through F draw FG par[l] to BA. [I. 31.
join AF.

Since ACE is an isosceles right-angled triangle, [Constr.
∴ each of the angles CAE, CEA is half a rt angle. [I. 32.
Similarly, each of the angles CBE, CEB is half a rt. angle.
∴ the whole angle AEB = a right angle.

Also, since GF is par[l] to AB,
∴ the angle EGF = a right angle. [I. 29.

And, since the angle GEF = half a right angle,
∴ the angle GFE = half a right angle [I. 32.

∴ EGF is an isosceles right-angled triangle.

Likewise, BDF is an isosceles right-angled triangle.

Now, the sum of the sqs on AE and EF
= the sq on AF [I. 47.
= the sqs on AD, DF. [I. 47.
= the sqs. on AD, DB

But the sq on AE = the sqs on AC and CE [I. 47
= twice the sq on AC.

Likewise, the sq on EF = twice the sq on GF
= twice the sq on CD [I. 34.

∴ twice the sum of the sqs on AC and CD
= the sum of the sqs. on AD and DB.

Therefore the squares on AD and DB are together double of the squares on AC and CD.

Wherefore, *if a straight line* &c. Q E D

EXERCISES

1 Divide a given straight line into two parts so that the sum of the squares on the two parts shall be the minimum (least possible)

2. The sum of the squares on any two lines is equal to twice the square on half their sum together with twice the square on half their difference

3. Divide a given straight line into two parts, so that the sum of the squares on the two parts may be equal to the square on another given line, the square on the second given line being greater than double the square on half the first line.

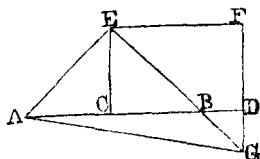
4 If a straight line be divided into any number of parts the sum of the squares on all the parts is a minimum when all the parts are equal

5 A point is taken in the hypotenuse of a right angled isosceles triangle, the square on the distance of this point from the vertex of the right angle is equal to half the sum of the squares on the segments of the hypotenuse

Proposition 10. Theorem

If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the sum of the squares on half the line bisected, and on the line made up of the half and the part produced

Let the straight line AB be bisected at C and produced to the point D

then the squares on AD, DB *shall be together double of the squares on* AC, CD

From C draw CE at right-angles to AB, [I 11.
and make CE equal to AC or CB, [I 3
join AE, EB
through E draw EF par[1] to AB, [I 31
and through D draw DF par[1] to CE meeting EF at F

Then, since CE and DF are par[1],
the sum of the angles CEF and DFE
=two rt angles. [I. 29.
the sum of the angles BEF and DFE
=less than two rt angles

∴ EB and FD will meet if produced [Ax. 12.

Let them meet at G, join AG

Now, it may be proved as in the last Prop.,
that each of the triangles ACE, BCE, EFG and BDG are isosceles right-angled triangles,
moreover that the angle AEG = a right angle.

Hence, the sum of the sqs. on AE and EG
= the sq. on AG [I. 47.
= the sqs. on AD, DG [I. 47.
= the sqs. on AD, DB.

But the sq. on AE = the sqs. on AC, CE [I. 47.
= twice the sq. on AC.

Likewise, the sq. on EG = twice the sq. on EF
= twice the sq. on CD. [I. 34.

∴ twice the sum of the sqs. on AC and CD
= the sum of the sqs. on AD and DB.

Therefore the squares on AD and DB are together double of the squares on AC and CD.

Wherefore, *if a straight line &c.* Q. E. D.

EXERCISES.

1. Produce a given straight line so that the squares on the whole line thus produced and the part produced shall be twice the square on the original line.

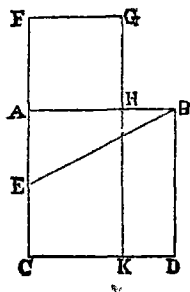
2. To produce a given straight line, so that the square on the produced part may be three the square on half the line.

3. To produce a given straight line, so that the sum of the squares on the whole line produced and on the produced part may be equal to the square on a second given line not less than the first.

4. Given the sum of the sides and the sum of the squares on the sides, to construct the rectangle.

5. The sum of the squares on the two sides of a triangle is equal to double the square on half the base, together with double the square on the line from the middle point of the base to the vertex.

Proposition 11. Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole line and one of the parts shall be equal to the square on the other part

Let AB be the given straight line it is required to divide it into two parts, so that the rectangle contained by the whole line and one of the parts shall be equal to the square on the other part

On AB describe the square ACDB , [I. 46.
bisect AC at E , [I. 10.
join BE ,
produce CA to F, and make EF = EB , [I 3.
and on AF describe the square AFGH.

Then AB *shall be divided at* H *so that the rectangle* AB, BH *is equal to the square on* AH

Produce GH to K

Then, since AC is bisected at E and produced to F,
∴ the rect CF FA with the sq on AE
= the sq on EF [II 6.
= the sq on EB [*Constr.*
= the sum of the sqs on AB and AE, [I. 47.
since EAB is a right angle

From these equals take away the common part, the sq. on AE :

∴ the rect. CF, FA = the sq on AB
= the fig AD. [*Constr.*

But the rect CF, FA = the fig. FK, since FG = FA,
∴ the fig. FK = the fig. AD.

From these equals take away the common part, the fig. AK
∴ the fig. FH = the fig HD
But the fig FH = the sq on AH, [Constr.
and the fig HD = the rect AB, BH since AB = BD
∴ the rect AB, BH is equal to the square on AH

Wherefore, *the straight line* AB *is divided at* H, *so that the rectangle* AB, BH *is equal to the square on* AH Q E F.

Def. A line divided, as in this Proposition, is said to be cut "*in extreme and mean ratio*"

EXERCISES

1 The straight line FC is cut in extreme and mean ratio at the point A

2 In figure to this proposition prove that the rectangle contained by AH, HB is equal to the rectangle contained by their sum and difference

3 If a straight line be divided so that the rectangle contained by the whole line and the smaller segment is equal to the square on the greater, the greater segment will be similarly divided by cutting off from it a part equal to the smaller segment

4 Produce a given straight line so that the square on the whole line thus produced may be equal to five times the square on the difference between the given line and the produced part.

5 If a straight line be divided "*in extreme and mean ratio*," the rectangle contained by the segments is equal to the difference of their squares

6 Join BF and CH, and produce CH to meet BF at L, show that CL is perpendicular to BF

7 If a straight line be cut "*in extreme and mean ratio*" the sum of the squares on the whole line and on the smaller part is equal to three times the square on the other part

8 If F, D, be joined cutting AHB and GHK in M and N respectively, shew that FM = DN

9 If a perpendicular from the right angle of a right-angled triangle on the hypotenuse divide it "*in extreme and mean ratio*," shew that the square on the side adjacent to the larger segment is equal to the rectangle contained by the hypotenuse and the other side

Proposition 12 Theorem

In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced

then the square on AB *shall be greater than the squares on* AC, CB, *by twice the rectangle* BC, CD.

Since BD is divided into two parts at C,

the sq on BD = the sum of the sqs. on BC, CD, together with twice the rect BC, CD [II 4.

To each equal add the sq on AD

the sum of the sqs on BD and AD = the sum of the sqs on BC, CD, AD, together with twice the rect BC, CD

But the sum of the sqs on BD and AD = the sq on AB, [I. 47.

and the sum of the sqs on CD and AD = the sq on AC .

the sq on AB = the sum of the sqs on BC, AC, together with twice the rect BC, CD

Therefore the square on AB is greater than the squares on AC, BC by twice the rectangle BC, CD

Wherefore, *in an obtuse-angled triangle* &c. Q. E. D.

EXERCISES

1 Each of the angles at the base of an isosceles triangle is double the vertical angle, shew that the square on any of the equal sides is equal to the square on the base together with the rectangle contained by any side and the base.

2 If each of the acute angles A and B of a triangle ABC be equal to half the angle of an equilateral triangle, then the square on AB is equal to three times the square on AC

3 If the sum of the two acute angles A and B in a triangle ABC be equal to an angle of an equilateral triangle, then,

$$AB^2 = AC^2 + BC^2 + AC \cdot BC.$$

4 In an obtuse angled triangle ABC if perpendiculars AD, CE be drawn from the acute angles A and C on the opposite sides produced, shew that the rectangle under AB, BE is equal to the rectangle under CB, BD

5 The sum of the squares on the diagonals of a trapezoid is equal to the sum of the squares on the two sides which are not parallel together with twice the rectangle under the sides which are parallel

6 The square on the line joining a point on a side of an equilateral triangle with the opposite vertical angle is equal to the sum of the squares on the segments of the side and the rectangle under those segments

Proposition 13 Theorem.

In every triangle, the square on the side subtending an acute angle, is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle

Let ABC be any triangle, and the angle at B an acute angle, and on BC one of the sides containing it, let fall the perpendicular AD from the opposite angle

then the square on AC, *opposite to the angle* B, *shall be less than the squares on* AB, BC, *by twice the rectangle* BC, BD

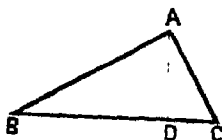
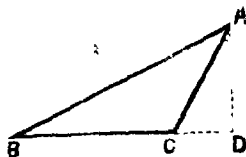



Fig 1 Fig 2

Now the perp AD may fall either within the triangle ABC as in Fig 1, or without the triangle as in Fig 2

Since, in Fig 1, BC is divided into two parts at D,
and, in Fig 2, BD is divided into two parts at C
in each figure,
the sqs on BC, BD, together = twice the rect BC, BD,
together with the sq on CD [II 7.

To each equal add the sq on AD

∴ the sum of the sqs on BC, BD, AD
= twice the rect BC, BD, together with the sqs on CD, AD.

But the sum of the sqs on BD, AD = the sq on AB, [I 47.
and the sum of the sqs on CD, AD = the sq on AC

∴ the sum of the sqs on BC, AB
= twice the rect BC, BD together with the sq on AC:

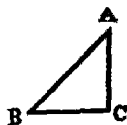
that is, the square on AC alone is less than the squares on AB, BC, by twice the rectangle BC, BD

If the perp AD coincide with the side AC:

Then BC is the straight line between the perpendicular and the acute angle at B, and the rect BC, BC = the sq on BC

Hence, it is manifest that the squares on AB, BC are equal to the square on AC, together with twice the square on BC

[I 47, and *Ax* 2

Wherefore, *in every triangle* &c Q.E D.

EXERCISES

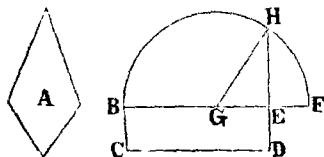
1 If a perpendicular to AC be drawn from the angle B, shew that the rectangle contained by the side AC and the part of it intercepted between this perpendicular and C, is equal to the rectangle contained by BC, BD

2 In any triangle the squares on the two sides are together double of the squares on half the base and on the straight line joining its bisection with the opposite angle

3 If two points be taken in the diameter of a circle, equally distant from the centre, and if straight lines be drawn from them to any point in the circumference, the sum of the squares on these lines is the same, whatever be the point

Proposition 14. Problem.

To describe a square that shall be equal to a given rectilineal figure

Let A be the given rectilineal figure : it is required to describe a square that shall be equal to A

Describe the rectangular par^{m} BCDE = the fig A [I 45

Then if the sides of it BE, ED are equal to one another, it is a square, and what was required is done

But if they are not equal produce one of them BE to F, and make EF equal to ED, bisect BF at G [I 10.

From centre G, with radius GB, or GF, describe the semicircle BHF,

and produce DE to meet the circumference at H

The square on EH shall be the required square

Join GH

The rect BE, EF, together with the sq on GE,
= the sq on GF [II 5
= the sq on GH
= the sum of the sqs on GE EH [I 47

∴ the rect BE, EF together with the sq on GE
= the sqs on GE, EH

Take away the sq on GE which is common to both
the rect BE, EF = the sq on EH

But the rect BE, EF is the par^{m} BD, since EF = ED [Cons

Therefore BD is equal to the square on EH

But BD is equal to the rectilineal figure A [Cons.

Therefore the square on EH is equal to the rectilineal figure A

Wherefore, *a square has been described equal to the given rectilineal figure* A, *namely the square described on* EH Q E F

EXERCISES

1 To describe a rectangle equal to a given square and having one of the sides equal to a given straight line

2 To cut a straight line externally so that the rectangle under the segments may be equal to the square on half the line.

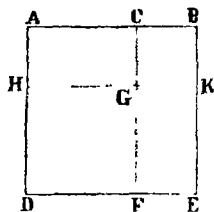
ALTERNATIVE PROOFS.

(Props. 4, 5, 6, 7, 8, 9, 10, 12 & 13)

Proposition 4 Theorem

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts

Let the straight line AB be divided into any two parts at C

Then the square on AB *shall be equal to the squares on* AC *and* CB, *together with twice the rectangle contained by* AC, CB.

On AB describe the square ADEB

From AD, cut off AH equal to CB

Then HD=AC, since AD=AB

Draw HK, CF, par[1] to AB, AD, respectively, intersecting each other at G

All the quadrilaterals are rectangles [I 29.

Also BK=AH=CB, [I 34

∴ the figure CK is equilateral

But the angle at B is a right angle,

∴ the other angles of CK are right angles [I 29, 34

Therefore CK is a square, and it is on CB

Likewise HF is a square and it is on HG, and HG=AC

Again, because GK=CG, and GF=HG,

∴ the rect GE=the rect AG

But rect AG=rect AC, CG,

and CG=CB,

∴ rect AG=rect AC, CB

∴ the rect AG with the rect GE=twice the rect AG,

=twice the rect AC, CB

Now the sq on AB=the fig AE

=the figs HF, CK AG, GE

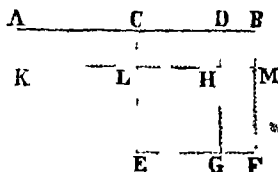
=the squares on AC CB, together with twice the rect AC, CB

∴ *the square on* AB=*the squares on* AC, CB, *together with twice the rect.* AC, CB Q E D.

Proposition 5 Theorem.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D

Then the rectangle contained by AD, DB, *together with the square on* CD, *shall be equal to the square on* CB.

On CB describe the square CEFB From CE cut off CL equal to DB Then LE=DC

Draw DG parl to CE or BF, meeting EF at G

Through L draw KLHM parl to AB, cutting DG at H, and meeting BF at M.

Draw AK parl to BF or CE, meeting MHLK at K

All the quadrilaterals are rectangles [I 29.

Because BF=CB=AC, and DB=CL=AK, [I 34

∴ rect AL=rect DF

Add rect CH to these equals

∴ rects AL, CH, *that is*, the rect AH=the rects CH, DF.

But rect AH=rect AD, DB, for DH=AK=DB

∴ rect. AD, DB=rects CH, DF

Also, LE=CD=LH,

∴ LG is a square, and it is on LH or CD

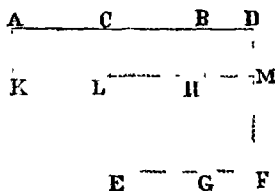
∴ rect AD, DB together with the square on CD
=rects CH, DF, with the sq LG
=the whole fig CF
=the sq on CB

∴ *rect* AD, DB *together with the square on* CD
=*the square on* CB. Q. E. D.

Proposition 6. Theorem.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced

Let the straight line AB be bisected at C, and produced to the point D

Then the rectangle AD, DB, *together with the square on* CB, *shall be equal to the square on* CD

On CD, describe the square CF

From CE cut off CL=BD, then LE=CB

Draw BG parl to CE or DF, meeting EF at G.

Through L draw KLM parl to AD, cutting BG at H, and meeting DF at M

Draw AK parl to CE or DF

All the quadrilaterals are rectangles. [I. 29.

Now, because DM=CL=BD,
and HG=LE=CB=AC,

∴ rect AL=rect HF Add to these equals rect. CM.
∴ rect AM=rects CM and HF.
But rect AM=rect AD, DB
∴ rect. AD, DB=rects CM and HF

Also, rect LG=square on LH=square on CB

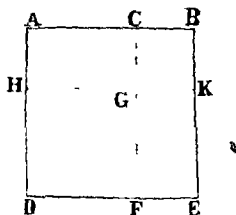
∴ rect AD, DB, together with the square on CB
=rects CM, HF, LG
=the whole fig CF
=the sq on CD.

∴ *rect* AD, DB, *together with the square on* CB
=*the square on* CD. Q. E. D.

Proposition 7. Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts, are equal to twice the rectangle contained by the whole line and that part, together with the square on the other part

Let the straight line AB be divided into any two parts at the point C :

then the squares on AB, BC *shall be equal to twice the rectangle contained by* AB, BC, *together with the square on* AC

Complete the figure as in Prop 4

All the quadrilaterals are rectangles [I 29.

Because CB = AH = BK,

∴ CK is a square and it is on CB

Likewise HF is a square and it is on HG, and HG = AC.

Again, BK = CB, and AB = BE

∴ rect AK = rect CE

∴ twice rect AK = rects AK, CE

But twice rect AK = twice rect AB, BC
and rects AK, CE = the fig AKF, with the sq CK

∴ twice rect AB, BC = the fig AKF, with the sq. CK.

Add to these equals the sq on AC, or sq HF.

∴ twice rect AB, BC together with the sq on AC
= the fig AKF together with the sqs HF, CK.

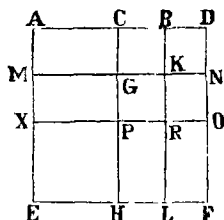
But the fig AKF, with the sq HF
= the whole fig AE = the sq on AB :

∴ *twice the rect* AB, BC, *together with the square on* AC
= *the squares on* AB *and* BC Q. E. D.

Proposition 8 Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and that part.

Let the straight line AB be divided into any two parts at the point C

then four times the rectangle contained by AB, BC, *together with the square on* AC, *shall be equal to the square on the straight line made up of* AB *and* BC *together*

Produce AB to D, so that BD=CB [*Post* 2 and I. 3.

On AD describe the square AEFD

From AE cut off AM and MX each equal to CB

Through C, B, draw CH BL par[l] to AE,
and through M X, draw MN, XO par[l] to AD,
cutting CH at G and P, and BL at K and R, respectively

All the quadrilaterals are rectangles [I. 29.

∵ XE=AC=XP,

∴ XH is a square, and it is on XP which is equal to AC

Also, ∵ XE=AC=MG=XP=PH=RL,
and CB or BD=CG=GP=PR=RO,

∴ the four rectangles AG, MP, PL, RF are equal,
and the four rectangles CK, BN, GR, KO are equal.

But the sum of these eight rectangles
=four times AG, CK
=four times AK
=four times the rectangle AB, BC.

∴ four times the rectangle AB, BC
=the fig AOH.

Add to these equals the square on AC or square XH ;

∴ four times the rect AB, BC, together with the sq. on AC
=the fig AOH, together with the sq XH
=the whole fig AF
=the sq on AD

∴ *four times the rect* AB, BC, *with the sq on* AC
=*the sq on the line made up of* AB *and* BC Q. E. D.

Props 9 and 10 may be included in one Proposition :—

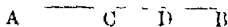
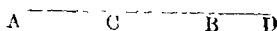
If a straight line be divided into two equal parts, and also into two unequal parts (either internally, or externally), the squares on the unequal parts are together double of the squares on half the line and on the line between the points of section

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D (internally as in Fig 1, and externally as in Fig 2)

FIG 1

A C D B

FIG 2

A C B D

Then the sqs on AD, DB=*twice the sqs on* AC, CD

For, sq on AD=sqs on AC, CD, with twice rect AC, CD [II 4.
Also, sq. on DB, with twice the rect. CD, CB=sqs on CD, CB. [II 7.

Add these equals, noting that AC=CB,

∴ sqs on AD, DB, with twice the rect AC, CD
=twice the sqs on AC, CD, with twice the rect AC, CD.

From these equals take away the common part twice rect AC, CD.

∴ *the sqs. on* AD, DB=*twice the sqs on* AC, CD. Q. E. D.

Proposition 12 Theorem

In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced :

then the square on AB *shall be greater than the squares on* AC, CB, *by twice the rectangle* BC, CD.

Describe squares AE, CK, CF, on the three sides.

Produce AD, KH to meet at N.

Draw BP perp. to AC produced.

Produce BP and FG to meet at M.

Draw CL par.[l] to BE.

As in I. 47, by joining FB and CO, we can prove that the rect. AM = the rect. AL.

Likewise, joining AK and CE, we can prove that the rect. BN = the rect. BL.

In like manner joining GB and AH we can prove that the $\triangle$GBC = $\triangle$CAH, [I. 4.

∴ twice these are equal.

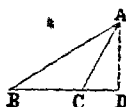
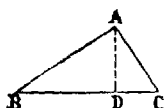
∴ the rect. CM = the rect. CN.

The square on AB
=rectangles AL, BL
=rectangles AM, BN
=the sqs. on AC, CB, with twice rect. CN
=the sqs. on AC, CB, with twice the rect. BC, CD. Q. E. D.

Proposition 13. Theorem.

In every triangle, the square on the side subtending an acute angle, is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle

Let ABC be any triangle, in which the angle ABC is acute; let AD be perpendicular to BC or BC produced

then the square on AC *shall be less than the squares on* AB, BC *by twice the rectangle* BC, BD

The sum of the sqs on BC, BD

=the sq on CD, with twice the rect BC, BD [II. 7.

Add to these equals the sq on AD

∴ the sum of the sqs on BC, BD, AD

=the sqs on CD, AD, with twice the rect. BC, BD

But the sqs on BD, AD=the sq on AB, [I 47

and the sqs on CD, AD=the sq on AC, [I. 47

∴ the sum of the sqs on AB, BC

=the sq on AC, with twice the rect BC, BD

Therefore the square on AC alone is less than the squares on AB, BC, by twice the rectangle BC, BD

Wherefore, *in every triangle the square* &c Q E. D.

ANOTHER PROOF.

Construct the figure as in the preceding Proposition.

As in the Alternative Proof of Proposition 12,

BL=BN, AL=AM, CM=CN

The square on AB=rects BL, AL

=rects BN, AM

=sqs on BC, AC, less twice the rect CN

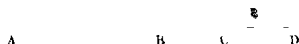
=sqs on BC, AC, less twice the rect BC, CD.

Q E D.

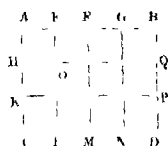
NOTES ON BOOK II.

We have said in the introduction to Book II that in Geometry lengths are not measured in terms of any such Arithmetical units as *feet*, *yards*, etc., nor areas in terms of *square feet*, *square yards*, etc.; but are measured by means of given lines, and rectangles contained by given straight lines, respectively. Nevertheless even if we were to give numerical values to straight lines and areas, *i. e.* measure them as in Arithmetic by means of a line of unit length (*e. g.* one foot), and a surface of unit area (*e. g.* one square foot), respectively, the Propositions established in Book II would yet be true; in fact they would be discovered to be nothing else than well-known algebraical formulas in disguise. We shall shew this presently.

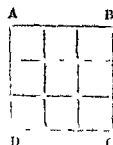
A line may be measured by another line, the length of which is arbitrarily assumed, and is called the unit of linear measurement. Thus a line AB may be measured by another line CD when the length of

A B C D

AB is represented by the number of linear units contained in it, the length CD being assumed as the linear unit.

A surface may be measured by another surface. A square, the side of which is one unit in length, is assumed for the measure of surfaces.

Let AB, equal to four linear units, and AC, equal to three, be placed at right angles. Complete the rectangle ABDC. Divide AB into four equal parts AE, EF, FG, GB, and AC into three equal parts AH, HK, KC. Through E, F, G, draw EL, FM, GN, parallel to AC or BD, and through H, K, draw HQ and KP parallel to AB or CD.

The rectangle AD is divided into four equal parts AL, FM, FN and GD. Therefore AD is four times AL; again AL is divided into three equal parts, AO, OK, and KL. Therefore AL is three times AO, which is the square on AE. Therefore the whole figure AD is divided into 3×4 or 12 equal squares, each of which is equal to the square on AE. Thus the area of the whole figure AD may be represented by the number 12, which is the product of the number of linear units in the sides.

A square described on a straight line which represents an inch is called a *square inch*. Thus if AB represent 4 inches, and AC 3 inches, then the whole rectangle AD is divided into 12 square inches. Therefore AD contains 12 *square inches*.

Let AB represent 3 inches. On AB describe a square ABCD Then by drawing straight lines parallel to the sides,* the whole figure may be divided into nine squares Therefore, AC is equal to 3×3 or 3^2 or 9 square inches.

It might now be seen that the area of a rectangle contained by two given straight lines *may be expressed* by the product of the two numbers which represent the lengths of these given straight lines in terms of the unit straight line, moreover, the area of a square described on a given straight line *may be expressed* by the square of the number which represents the length of the given straight line in terms of the unit straight line But it must clearly be understood that such expressions are rigorously excluded from Geometry proper, as they involve Arithmetical or Algebraical methods, *and are only used as illustrations*, to shew that the Geometrical truths of Book II hold good in any particular case we might take

For the sake of brevity, however, the rectangle contained by AB, BC may sometimes be expressed as a product, thus AB BC, and the square described on AB as AB^2 But it must be observed most carefully that such a notation, besides the ordinary signs + and −, when used in Geometry proper, are merely verbal abbreviations, and do not imply any Arithmetical or Algebraical formula

Prop. 1 *Algebraical Illustration* (Fig Prop 1)

Let BC be represented by a linear units, and the line A by b linear units of the same kind Also let the parts BD, DE and EC be represented by m, n, p linear units

Then $a=m+n+p$. Multiply these equals by b,

therefore $ab=mb+nb+pb$

Therefore the number of square units expressed by the product ab is equal to the number of square units expressed by the sum of the products mb, nb, pb

Area of a Triangle

The area of a triangle is equal to half that of the rectangle, one of whose sides is the same as the base of the triangle, and the other side the altitude of the triangle. (Euc I 41)

Proposition 2 is only a corollary to Proposition 1, for, when the two straight lines mentioned in its enunciation are equal, then the rectangle contained by them is a square The square on AB is equal to the rectangle contained by AB and AD, but AB is divided into two parts AC, CB Therefore the square on AB is equal to the rectangle contained by AB, AC, together with the rectangle contained by AB, CB. (Prop 1)

Prop 2 Algebraical Illustration (Fig Prop 2)

Let AB, AC and CB be represented by a, m and n respectively; then $m+n=a$ Multiply these equals by a, therefore $am+an=a^2$.

Wherefore the sum of the products of the whole a, and each of the parts m, n, is equal to the square of the whole line a

PROP 3 may be deduced from Proposition I, the undivided line in which being equal to one of the two segments of the divided line.

Prop 3 Algebraical Illustration (Fig Prop 3)

Let AB, AC, CB contain a, m and n linear units respectively;

then $a=m+n$ multiply these equals by n,

$\therefore an=mn+n^2$

Wherefore if a number representing a line be divided into any two parts, the product of the whole number and one of the parts, is equal to the product of the two parts together with the square of the aforesaid part

Prop 4 may be proved without drawing any diagram

The sq on AB=the rect AB AC+the rect AB, BC [II 2

But the rect AB, AC=the sq on AC+the rect AC, CB [II 3

And the rect AB, BC=the sq on BC+the rect AC, CB [II 3

$\therefore$ the sq on AB

=the sq on AC+the sq on BC+twice the rect AC, CB Q E D

Prop 4 Algebraical Illustration (Fig Prop 4)

Let AB, AC, CB represent a, m and n respectively,

then $a=m+n$, squaring these equals,

$a^2=(m+n)^2=m^2+n^2+2mn$

Wherefore if a number representing a line be divided into any two parts, the square of the number is equal to the squares of the two parts, together with twice the product of the two parts

Prop 5 may also be proved in the following manner —

The rect AD, DB=the rect AC, DB+the rect CD, DB [II 1

=the rect CB, DB+the rect CD, DB,

since AC=CB

But the rect CB, DB=the sq. on DB+the rect. CD, DB [II. 3.

$\therefore$ the rect AD, DB=the sq on DB+twice the rect. CD, DB.

Add the sq on CD to these equals,

∴ the rect AD, DB+the sq on CD
=the sq on CD+the sq on DB+twice the rect. CD, DB.
=the sq on CB [II. 4.

Prop 5 Algebraical Illustration (Fig Prop 5)

Let $AB=2a$, $CD=n$, then $AC=CB=a$, and $DB=CB-CD=a-n$,
also $AD=a+n$

$$(a+n)(a-n)=a^2-n^2$$
$$(a+n)(a-n)+n^2=a^2$$

Prop 6 may be proved without making any geometrical construction and the preceding Proof of Prop 5 may be applied

Prop 6 Algebraical Illustration (Fig Prop 6)

Let $AB=2a$, $CD=n$ then $AC=CB=a$, $BD=n-a$,
$AD=CD+AC=n+a$

$$(n+a)(n-a)=n^2-a^2$$
$$(n+a)(n-a)+a^2=n^2$$

Observing the remark in the Introduction to Book II about the *internal* and *external* division of a straight line, Props 5 and 6 may be expressed in one enunciation, thus —

If a straight line be divided into two equal, and also into two unequal segments (either internally or externally), the rectangle contained by the two unequal segments is equal to the difference of the squares on half the line and on the line between the points of section

It might be seen that this joint enunciation is only a particular case of the following general proposition —

The rectangle contained by two straight lines, together with the square on half their difference, is equal to the square on half their sum that is —*The rectangle contained by the sum and difference of two straight lines, is equal to the difference of their squares*

Prop 7 may be proved without any geometrical construction

The sq on AB=the sqs on AC, CB, with twice the rect AC, CB.
[II 4

To these equals add the sq on BC

∴ the sqs on AB, BC=the sq on AC+twice the sq. on BC +twice the rect AC, CB

But the sq on BC+the rect AC, CB=the rect. AB, BC
[II. 3.

∴ the sqs on AB, BC=the sq on AC+twice the rect. AB, BC.

Prop 7. *Algebraical Illustration* (Fig Prop 7).

Let AB$=a$, BC$=b$, AC$=d$

$\therefore\ d=a-b$

$\therefore\ d^2=(a-b)^2$

$=a^2+b^2-2ab$

$\therefore\ d^2+2ab=a^2+b^2$

Prop 8 *may also be proved without any geometrical construction.*

Produce AB to D, making BD=BC, then AD=AB+BC

The sq on AD=the sqs on AB, BD+twice the rect AB, BD [II 4

=the sqs on AB, BC+twice the rect AB, BC

But the sqs on AB, BC=twice the rect AB, BC
+the sq on AC [II 7.

∴ the sq on AD=four times the rect AB, BC+the sq on AC.

Prop 8 *Algebraical Illustration* (Fig Prop 8)

Let AB$=x$, CB$=y$, AD$=a$, AC$=d=$AB$-$BC$=x-y$,

$a=x+y,\ d=x-y,$

$a^2=x^2+2xy+y^2,$

$d^2=x^2-2xy+y^2;$

$\therefore\ a^2-d^2=4xy,$

$\therefore\ a^2=4xy+d^2$

Prop 9 *Algebraical Illustration*

Let AB be divided equally at C and unequally at D

A —— C —— D —— B

AC=CB$=x$ suppose, then AB$=2x$

Let CD$=a$, and DB$=b$, then $a+b=x$

AD$=x+a$, and $b=x-a$,

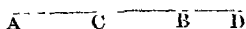
$(x+a)^2+b^2=(x+a)^2+(x-a)^2$

$=x^2+2ax+a^2+x^2-2ax+a^2$

$=2x^2+2a^2.$

Prop 10 Algebraical Illustration.

Let AB be bisected at C and produced to D

A C B D

AC=CB=x suppose, then AB=$2x$

Let CD=a, and DB=b, then $a=b+x$ AD=$x+a$,

also $b=a-x$

$$(x+a)^2+b^2=(x+a)^2+(a-x)^2$$
$$=x^2+2ax+a^2+a^2-2ax+x^2$$
$$=2x^2+2a^2$$

The joint enunciation of Props 9 and 10 given in the Alternative Proof page 172, is only a particular case of the following general proposition —

The square on the sum of two straight lines and the square on their difference are together equal to double the sum of the squares on the two lines

In Prop 11 the division may be made externally as well as internally

H A H B

Thus BA may be produced to H so that the rect AB, BH [illegible] the sq on AH

In this case AB is also divided "in extreme and mean ratio" at H

In making the construction for this case we proceed as in Prop 11, but instead of producing EA to F, we produce EC to F, so that EF=EB On AF describe the square AFG'H', on the side remote from BD Produce DC to meet H'G' at K Now it may be proved as in Prop 11, that the rect AB, BH'=the sq on AH'

Prop 11 Algebraical Illustration

To find the point H in AB such that AB BH=AH^2.

Let AB=a, and AH one of the unknown parts=x.

Then the other part HB=$a-x$

Then $a(a-x)=x^2$, by the problem,

$\therefore x^2+ax=a^2$

Add $\frac{1}{4}a^2$ to both sides of this equation.

Then $x^2+ax+\frac{1}{4}a^2=a^2+\frac{1}{4}a^2=\frac{5}{4}a^2$

$\therefore\ (x+\frac{1}{2}a)^2=\frac{5}{4}a^2$

Extracting the square root of both sides,

$$x+\tfrac{1}{2}a=\pm\tfrac{1}{2}a\sqrt{5}$$

$\therefore\ x=\frac{1}{2}a\sqrt{5}-\frac{1}{2}a$, taking the upper sign

$=\frac{1}{2}(\sqrt{5}-1)a$

Also, $x=-\frac{1}{2}a\sqrt{5}-\frac{1}{2}a$, taking the lower sign

$=-\frac{1}{2}(\sqrt{5}+1)a$

It will be seen that by taking the upper sign we get the *internal* point H; also by taking the lower sign, we get the *external* point H′; for, in the second case the negative value of x means that the point H must be *on that side of* A *which is remote from* AB, that is, *in* BA *produced*.

The expressions for x obtained algebraically, correspond with the geometrical construction; for, taking only the internal section at H:—

$$EB=EF=EA+AF=\tfrac{1}{2}a+x$$

$\therefore\ (EB)^2=(\frac{1}{2}a+x)^2$

Again $(EB)^2=(AE)^2+(AB)^2=(\frac{1}{2}a)^2+a^2=\frac{1}{4}a^2+a^2=\frac{5}{4}a^2$

$\therefore\ (\frac{1}{2}a+x)^2=\frac{5}{4}a^2$

The student might prove the other case as an exercise.

Prop. 12. Algebraical Illustration.

Let $BC=a$, $CA=b$, $AB=c$, $CD=m$, $DA=n$.

Then $BD=a+m$

$c^2=(a+m)^2+n^2$ [I. 47.

$b^2=m^2+n^2$ [I. 47.

Subtracting $c^2-b^2=(a+m)^2-m^2$

$=a^2+2am+m^2-m^2$

$=a^2+2am$

$\therefore\ c^2=a^2+b^2+2am$

Prop. 13. Algebraical Illustration.

Let $BC=a$, $CA=b$, $AB=c$, $BD=n$, and $AD=m$.

$DC=a-n$ or $n-a$, in Fig. 1 or Fig. 2.

$(a-n)^2=(n-a)^2$

$c^2=m^2+n^2$

$b^2=m^2+(a-n)^2$ or $=m^2+(n-a)^2$

$$\therefore\ c^2-b^2=n^2-(a-n)^2 \text{ or } =n^2-(n-a)^2$$
$$=n^2-a^2-n^2+2an$$
$$=-a^2+2an$$
$$\therefore\ c^2+a^2=b^2+2an$$

Using the definition of orthogonal projection, page 51, we might enunciate Props 12 and 13 together thus:—

In any triangle the square on one of the sides is equal to the sum of the squares on the other two sides ± twice the rectangle contained by one of these sides and the projection of the other side upon it, the upper or lower sign being taken accordingly as in the triangle the first side subtends an obtuse or an acute angle.

By the help of Prop 13, we may deduce the area of a triangle in terms of the sides. For

$$b^2+2an=c^2+a^2,$$
$$2an=c^2+a^2-b^2 \qquad \text{(I)}$$
$$\text{and } 2ac=2ac \qquad \text{(II)}$$

Adding (I) and (II),

$$2a(c+n)=c^2+a^2+2ac-b^2=(c+a)^2-b^2$$
$$=(c+a+b)(c+a-b) \qquad \text{(A)}$$

Subtracting (I) from (II),

$$2a(c-n)=b^2+2ac-a^2-c^2=b^2-(a-c)^2$$
$$=(b+a-c)(b-a+c) \qquad \text{(B)}$$

Multiplying (A) by (B),

$$4a^2(c^2-n^2)=(c+a+b)(c+a-b)(a+b-c)(b+c-a)$$

If $2s$ be the perimeter of the triangle,

then $2s=a+b+c$

$2s-2a=b+c-a$

$2s-2b=a+c-b$

$2s-2c=a+b-c$

$$\therefore\ 4a^2(c^2-n^2) \text{ or } 4a^2m^2 \quad (\text{because } m^2=c^2-n^2)$$
$$=2s(2s-2a)(2s-2b)(2s-2c)$$
$$\therefore\ \tfrac{1}{4}a^2m^2=s(s-a)(s-b)(s-c)$$
$$\therefore\ \tfrac{1}{2}am=\sqrt{\{s(s-a)(s-b)(s-c)\}}$$

But $\frac{1}{2}am$ = area of the triangle

$\therefore$ area of the triangle $=\sqrt{\{s(s-a)(s-b)(s-c)\}}$

———o———

QUESTIONS ON BOOK II

1. Define a *rectangle*

2 When does a *rectangle* become a *square*?

3 From II 1, deduce the following rule for finding the area of a triangle —Multiply any side by the perpendicular upon it from the opposite angle and take half the product

4. Show that the area of a rhombus is equal to half the rectangle contained by the diagonals

5 Find the area of a rectangle, two of whose adjacent sides are given

6 Show that the area of a four sided figure, two of the opposite sides of which are parallel, is equal to the rectangle contained by half the sum of the parallel sides and the distance between them

7 In II 4, if the two complements be together equal to the two squares, the given line is bisected

8 Prove II. 4, without drawing any diagram

9 Prove from II 4, how x^2+2xy may be made a complete square

10. Prove by II 4, that the square on a whole line is four times the square on half the line

11. Prove that the square on a whole line is nine times the square on a third part of it

12 Render the expression c^2+8c a perfect square

13 Prove from any proposition of Book II, that the product of the sum and difference of two straight lines is equal to the difference of their squares

14 What is meant by a straight line AB being cut externally at the point C?

15 When is the rectangle contained by the two parts of a straight line the greatest possible?

16 Prove II 5 without making any geometrical construction

17 If the sum of two straight lines be 39 inches, and the difference 9 inches, find the lines

18 Cut a line, whose length is 25 inches, internally, so that the rectangle under the segments shall be 150 square inches

19 Cut the same line externally, so that the rectangle under the segments shall be 600 square inches.

20. Cut a line, whose length is 100 feet, internally, so that the difference of the squares of the segments shall be 5,000 square feet.

21 Cut the same line externally, so that the difference of the squares of the segments shall be 14,000 square feet

22 Prove II. 7 without making any geometrical construction.

23 When is the sum of the squares on the two parts of a straight line the least possible? Prove this when the line is 20 inches long

24. Prove II 9 without making any geometrical construction

25. Include the enunciation of II 10 in that of II 9.

26 In fig II 11, how many similar points of section may be obtained?

27 In II 11, the larger segment is equal to 10 ($\sqrt{5}-1$), find the length of the whole line

28 The sides of a triangle are 13, 14 and 15 inches, find the segments of the side 14 inches long, made by a perpendicular from the opposite angle

29 The sides of a triangle are 3, 4, and 6 feet, shew that the triangle is obtuse angled, and if a perpendicular be drawn from the obtuse angle to the opposite side, find the length of each segment into which that side is divided by the perpendicular

30 Find the area of a triangle whose sides are 39, 41, and 50 feet

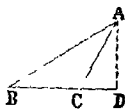
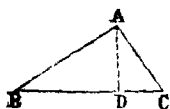
31 If the sides of a triangle be 12, 21, and 25 feet, find the lengths of the segments into which the perpendicular divides the side whose length is 21 feet Also find the length of the perpendicular

32 If the sides of a rectangle be 32 feet and 8 feet, respectively, calculate the side of the square equal to it

ADDITIONAL PROPOSITIONS ON BOOK II.

Proposition I Theorem

In any triangle, if a line be drawn from the vertex perpendicular to the base or base produced, the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.

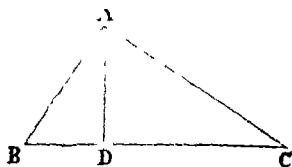
Let ABC be a triangle, let AD be perpendicular to BC or BC produced

The difference of the squares on AB, AC *is equal to the difference of the squares on* BD DC

The sq on AB = the sqs on AD BD, [I 47.
and the sq on AC = the sqs on AD DC
∴ the difference of the sqs on AB, AC
= the difference of the sqs on BD, DC Q E D

COR The rectangle contained by the sum and difference of the sides AB, AC is equal to the rectangle contained by the sum and difference of BD, DC [I 5 *Cor.*

Proposition II Theorem

In a right angled triangle if a perpendicular be dropped from the vertex of the right angle on the hypotenuse, the square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse

Let ABC be a right angled triangle, the angle at A being a right angle, let AD be perpendicular to BC

The sq on AD = *the rect* BD, DC
The sqs on BD, DC, with twice the rect BD, DC
= the sq on BC [II 4.

=the sqs on AB AC [I. 47.

=the sqs. on BD, DC, with twice the sq on AD,

∴ twice the rect BD, DC=twice the sq on AD

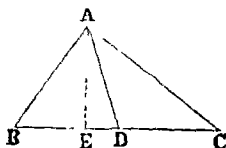
∴ the sq on AD=the rect BD, DC Q E D.

COR. In a right angled triangle if a perpendicular be dropped from the vertex of the right angle on the hypotenuse, the square on either side is equal to the rectangle contained by the projection of that side on the hypotenuse and the hypotenuse

AD^2=the rect BD, DC Add to these equals the square on BD; then apply I 47 and II 3

Proposition III Theorem

The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the median to the base

Let ABC be the triangle, D the middle point of the base BC,

then the sqs on AB, AC=*twice the sqs on* BD, AD

Draw AE perp to BC

In △ ADC,

the sq on AC=the sqs on AD, DC,

with twice the rect ED, DC [II 12.

In △ ADB,

the sq on AB=the sqs on AD, BD

less twice the rect ED, DB [II 13

But BD=DC

∴ the sqs on AB, AC=twice the sqs on BD, AD.

Q. E. D.

EXERCISES

1 The sum of the squares on the four sides of a parallelogram is equal to the sum of the squares on the diagonals.

2 The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the two diagonals together with four times the square on the line joining the middle points of the diagonals.

Proposition IV Theorem

The squares on the diagonals of a trapezoid are equal to the squares on the two sides which are not parallel, together with twice the rectangle contained by the parallel sides.

Let ABCD be the trapezoid and let AD, BC be the parallel sides. Join AC, BD.

The squares on AC, BD *will be equal to the squares on* AB, CD, *together with twice the rectangle contained by* AD, BC.

Of the two parallel sides let AD be greater than BC; from the points A, D, draw AE and DF perpendiculars to BC produced.

∴ $AC^2 = AB^2 + BC^2 + 2CB.BE$, [II. 12.

and $BD^2 + BC^2 = DC^2 + 2CB.BF$, [II. 13.

∴ by adding these equals

$$AC^2 + BD^2 + BC^2 = AB^2 + DC^2 + BC^2 + 2CB.BE + 2CB.BF.$$

∴ taking away the common part BC^2,

$AC^2 + BD^2$

$= AB^2 + DC^2 +$ twice the sum of the rects. CB.BE and CB.BF

$= AB^2 + DC^2 +$ twice the rect. CB.EF [II. 1.

$= AB^2 + DC^2 +$ twice the rect. CB.AD.

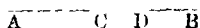
Wherefore, *the squares on the diagonals* &c. Q. E. D.

Maxima and Minima.

Proposition V Theorem

*The rectangle contained by the two segments of a straight line is the greatest possible when the two segments are equal**

Let AB be the given straight line, C its middle point and D any other point in AB

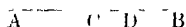
A C D B

The rectangle AC, CB or the square on CB is greater than the rectangle AD, DB by the square on CD [II 5

∴ the rectangle AC, CB is the greatest possible Q E D

Proposition VI Theorem

The sum of the squares on the two segments of a straight line is the least possible when the two segments are equal.*

Let AB be the given straight line, C its middle point and D any other point

A C D B

The sum of the squares on AC, CB or twice the square on AC is less than the sum of the squares on AD, DB by twice the square on CD [II 9

∴ the sum of the squares on AC, CB is the least possible

Q E D.

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* Beginners may omit the corresponding Algebraical illustrations of Props V and VI respectively —

If $a+b=$ constant,

then $ab=$ maximum

when $a=b$

Also, if $a+b=$ constant,

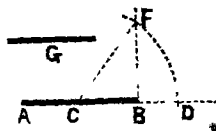
then $a^2+b^2=$ minimum

when $a=b$.

GEOMETRICAL EXERCISES ON BOOK II.

Proposition 1 Problem

To produce a given straight line so that the rectangle contained by the whole line produced, and the part produced may be equal to a given square

Let AB and G be the two given straight lines, it is required to produce AB so that the rectangle contained by the whole line produced and the part produced may be equal to the square on G.

ANALYSIS—Suppose AB is produced as required, and that BD is the produced part, bisect AB at C

∵ AB is bisected at C and produced to D,

∴ AD.DB+CB²=CD² [II 6

∴ AD.DB=CD²—CB²

But AD.DB=the sq on G, by supposition,

∴ the sq on G=CD²—CB²

∴ the sq on G+CB²=CD²

Hence we obtain the following clue, viz to produce CD so that CD be equal to the hypotenuse of the right-angled triangle whose sides are G and CB

SYNTHESIS—Bisect AB at C, and draw BF ⊥ to AB.

Make BF=G, and join CF

With centre C and radius CF describe a circle to intersect CB produced at D

Then, AD.DB+CB²=CD² [II 6.

=CF²

∴ AD.DB=CF²—CB² [I. 47.

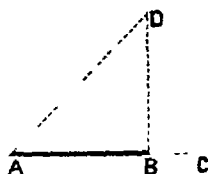
=BF²

=the sq on G

Wherefore AB is *produced to* D *so that* &c Q. E. F.

Proposition 2. Problem.

To produce a given straight line so that the square on the whole line thus produced shall be double of the square on the given straight line.

Let AB be the given straight line It is required to produce AB so that the square on the whole line thus produced shall be double of the square on AB

ANALYSIS —Suppose that AB is produced to C in the required manner

Then the sq on AC=double of the sq on AB

But we know that if we draw BD ⊥ to AB, and make BD=AB,

then $AD^2=2AB^2$ [I 47

∴ $AC^2=AD^2$

∴ $AC=AD$.

Hence we obtain the clue to the —

SYNTHESIS —Draw BD ⊥ to AB, and make BD=AB

Join AD, and produce AB to C so that AC=AD

∴ $AC^2=AD^2$

$=2AB^2$

From the above investigation we may easily deduce the method of producing a given straight line so that the square on the whole line thus produced may be 3, 4, 5, 6, 7, 8, etc., times respectively, the square on the original line.

If the number proposed be a perfect square, then it is obvious that the whole line must be that number of times of the original line which is the square root of the proposed number; *e g* if the proposed number is 9, then the whole line must be 3 times the original line

But if the number proposed is not a perfect square, then we must proceed thus —

Let it be asked *to produce a given straight line so that the square on the whole line thus produced shall be 21 times the square on the given line*

Let AB be the given line

A————————B

Now, take the nearest perfect square to 21 which is less than 21, viz 16

$$\sqrt{16}=4.$$

$$21-16=5$$

The nearest perfect square to 5, which is less than 5, is 4

$$\sqrt{4}=2$$

$$5-4=1$$

Now, describe a right-angled $\triangle$ with AB as one side, and 2AB as the adjacent side

$\therefore$ the sq on the hypotenuse of this $\triangle$

$=(1+4)$ times the sq on AB, [I 47.
$=5AB^2$

Again, describe another right-angled $\triangle$ with the above hypotenuse as one side, and 4AB as the adjacent side

$\therefore$ the sq on the hypotenuse of this last $\triangle$

$=(5+16)$ times the sq on AB, [I 47.
$=21\ AB^2$

Hence we must produce AB so that the whole line shall be equal to the hypotenuse of the last $\triangle$ Q E F.

MISCELLANEOUS EXERCISES ON BOOK II.

1. ABC is a right-angled triangle, the angle at C being a right angle. From AB cut off AD equal to AC, and from BA cut off BE equal to BC Prove that the square on DE is equal to twice the rectangle contained by BD, EA

2 To divide a given straight line into two such parts that the difference of their squares shall be equal to twice their rectangle.

3 From AC, the diagonal of a square ABCD, cut off AE equal to one fourth of AC, and join BE, DE Shew that the figure BADE equals twice the square on AE

4 The square on the hypotenuse of a right angled triangle together with four times the area of the triangle is equal to the square on the sum of the sides

5 If from a point in the base of an isosceles triangle a straight line be drawn to the opposite angle, the square on this line shall be less than the square on one of the equal sides of the triangle by the rectangle contained by the segments of the base.

6 Produce one side of a triangle so that the rectangle contained by it and the produced part may be equal to the difference of the squares on the other two sides.

7 In a triangle whose vertical angle is a right angle, a straight line is drawn perpendicular to the base, show that the square on this perpendicular is equal to the rectangle contained by the segments of the base

Also, shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side

8 Divide a given straight line into two parts, so that the rectangle contained by the two segments shall be equal to the square on any line less than half the given line

9 Produce a given straight line so that the rectangle contained by the whole line produced and the original line shall be equal to a given square, the side of which is greater than the given straight line

10 On a given straight line describe a rectangle which shall be equal to the difference of the squares on two given straight lines, any two of the three given lines being together greater than the third

11 Given a square and one side of a rectangle which is equal to the square, find the other side.

12 Divide a given straight line into two parts, so that the square on one part is double the square on the other part.

13. Produce a given straight line so that the rectangle contained by the whole line produced and the produced part shall be equal to a given square.

14 Produce a given straight line so that the rectangle contained by the whole line produced and the original line shall be equal to the square on the produced part

15 If an angle of a triangle be two thirds of a right angle, the square on the side opposite to it is equal to the sum of the squares on the sides containing it, diminished by the rectangle contained by them.

16. Prove that the square on any straight line drawn from the vertex of an isosceles triangle to the base, is less than the square on a side of the triangle by the rectangle contained by the segments of the base

17 If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are equal to twice their rectangle together with four times the square on the line between the points of section

18 On the radius of a circle a semicircle is described and from any point in this radius a straight line is drawn at right angles to it, cutting the smaller circle in P and the larger in Q If A be the common extremity of their diameters, then show that the square on AQ will be double the square on AP

19 Divide a straight line into two such parts that the squares on the whole line, and on one of the parts, shall be equal to twice the square on the other part

20 Show that in a straight line divided as in II 11, the rectangle contained by the sum and difference of the parts may be equal to the rectangle contained by the parts

21 Given the base and the difference of the squares on the sides of a triangle, show that the vertex will always lie on a straight line perpendicular to the base that is, show that the locus of the vertex is a straight line perpendicular to the base

22 Divide a straight line into two parts, so that the square on the greater part shall be equal to twice the rectangle contained by the whole line and the smaller part

23 If a perpendicular be drawn from either of the equal angles of an isosceles triangle on the opposite side, twice the rectangle contained by that side, and the part of it intercepted between the foot of the perpendicular and the base is equal to the square on the base

24 If an angle of a triangle be two thirds of two right angles, the square on the side subtending that angle is equal to the squares on the sides containing it, together with the rectangle contained by those sides

25 If in fig Euc I 47 the angular points be joined, the sum of the squares on the six sides of the figure so formed is equal to eight times the square on the hypotenuse.

26 In fig Euc II 11 join BF and CH, and produce CH to meet BF in L, and show that CL is perpendicular to BF

27. Describe an isosceles obtuse-angled triangle, such that the square on the side subtending the obtuse angle may be three times the square on either of the sides containing the obtuse angle

28 If AB, one of the equal sides of an isosceles triangle ABC, be produced beyond the base to D, so that BD be equal to AB, show that the square on CD is equal to the square on AB together with twice the square on BC.

29 The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals

30 If from the angles of a triangle lines be drawn bisecting the opposite sides, four times the squares on these lines is equal to three times the squares on the sides of the triangle

31 The sum of the squares on the sides of a right angled triangle is three times the sum of the squares on the sides of the triangle formed on the hypotenuse, by joining the points of trisection with the right angle

32 The squares on the diagonals of a trapezium are together less than the squares on the four sides by four times the square on the line joining the points of bisection of the diagonals

33 If two points C, D be taken on a straight line AB, then $AC \cdot DB + CD \cdot AB = AD \cdot CB$

34 In any triangle ABC, the angles of which at B and C are acute, if BD and CE be drawn perpendicular to AC, AB produced if necessary then shall the square on BC be equal to the rectangle under AB, BE together with the rectangle under AC, CD

35 If AD be drawn from the vertex A of the triangle ABC to cut BC at D such that $m \cdot BD = n \cdot CD$, then $m(AB)^2 + n(AC)^2 = m(BD)^2 + n(CD)^2 + (m+n)(AD)^2$

36 In any quadrilateral figure, a point E is taken, show that the squares on the lines drawn from E to the angular points of the figure, are together equal to the squares on half of each of the sides of the figure, together with the squares on the lines joining E with the middle points of those sides

37 If ABC be a triangle right-angled at A, and BE and CF be drawn bisecting the opposite sides respectively, show that four times the sum of the squares on BE, CF is equal to five times the square on BC

38 Construct a rectangle equal to a given square, and having the difference of its sides equal to a given straight line

39. The squares on the diagonals of any quadrilateral are together double of the squares on the two lines joining the points of bisection of the opposite sides

40 In a straight line find two points equally distant from its extremities, so that the square on the middle part shall be equal to the sum of the squares on the extreme parts.

41. If from any point within a rectangle, lines be drawn to the angular points, the sums of the squares on those drawn to the opposite angles shall be equal

42 Of a right-angled triangle, given one side and the adjacent segment of the hypotenuse made by the perpendicular from the right angle · construct the triangle

43 Given the area and the difference of the sides of a right-angled triangle construct it

44 Given the segments of the hypotenuse made by the perpendicular from the right angle; construct the right angled triangle.

45. Given the base of any triangle, the area, and the line bisecting the base construct the triangle

46 ABC is an equilateral triangle, and from B, BE is drawn at right angles to BA, and from A a perpendicular is drawn to BC and produced to meet BE at E Through E a straight line EF is drawn parallel to BA meeting BC in F, and through F a straight line FG is drawn perpendicular to AB meeting AB in G If AG be bisected in K, prove that the straight line AB is trisected in K, G

47 Divide a given straight line into two parts, so that the sum of the squares on the whole line and on one part, may be equal to three times the square on the other part

48 Given the base and the sum of the squares on the sides of any triangle, find the locus of the vertex

49 Given the hypotenuse describe a right angled triangle, so that the hypotenuse and one side may be together double of the third side

50 From the hypotenuse of a right angled triangle, portions are cut off equal to the adjacent sides, show that the square on the middle segment is equal to twice the rectangle under the extreme segments

51 If squares be described on the sides of any triangle, and the angular points of the squares be joined the sum of the squares on the sides of the hexagonal figure thus formed is equal to four times the sum of the squares on the sides

52 The base of a triangle is given and is bisected by the centre of a given circle if the vertex be at any point of the circumference, show that the sum of the squares on the two sides of the triangle is invariable.

53 Produce a given straight line so that the sum of the squares on the given straight line and on the part produced, may be equal to twice the rectangle contained by the whole straight line thus produced and the part produced

54. Produce a given straight line so that the square on the whole line thus produced may be double the square on the part produced

55 If ABC be a triangle, in which C is a right angle, and DE be drawn from a point D in BC at right angles to AB, prove that the rectangles contained by AB, AE and AC, AD will be equal.

HINTS FOR SOLUTION.

BOOK II.

Prop 3

2 Make the produced part equal to the given line.

Prop 4

5 See Addl Prop. II page 186

6 From AC the hypotenuse AD is cut off $= AB$ and CE is cut off $= BC$

$$AC^2 = AD^2 + DC^2 + 2AD \cdot DC,$$

again, $AC^2 = AD^2 + EC^2$

$$= AD^2 + ED^2 + DC^2 + 2ED \cdot DC, \text{ \&c}$$

Prop 5

3 Bisect the line

4 The difference of the squares on the sides is equal to the difference of the squares on the segments &c

5 Let AD be drawn to D a point in the base BC Draw $AE \perp BC$

$AB^2 - AD^2 = EB^2 - ED^2 = CD \cdot DB$

6 Let AB be the given line, bisect it at C, draw CD at right angles to AB making it equal to the given line

With C as centre and CA or CB as radius describe a ⊙ CD must not be greater than AC (Addl Prop V, page 189) If CD be equal to AC D will fall on the circumference and $CD^2 = AC \cdot CB$ If CD be less than AC, draw $DE \parallel AB$ meeting the circumference at E Draw $EF \perp AB$ F is the required point $EF^2 = CE^2 - CF^2 = CB^2 - CF^2 = AF \cdot FB + CF^2 - CF^2 = AF \cdot FB$

7. Let ABC be a triangle, AD perpendicular to BC, and E the middle point of BC The difference of the squares on AB, AC is equal to the difference of the squares on BD, DC, etc

Prop 6

2 See Prop 5, *Cor*

3 Let the given straight line AB be bisected at C and let it be produced to D making BD equal to AC or CB

4. See Prop 1, page 190

6. Proceed as in Ex. 5, Prop 5

Prop 7.

1 Let AB be the given line. Make the angle BAC equal to half a right angle and also make the angle ABC equal to one-fourth of a right angle. Make the angle BCD equal to the angle ABC, CD meeting AB at D. The square on AD is double of the square on DB. $AB^2+AD^2=2AB\cdot AD+BD^2$, $\therefore AB^2+2BD^2=2AB\cdot AD+BD^2$

Prop 9

1 See Addl Prop VI, page 189

2 Let AD, DB in the Fig be the two lines. $AD-DB=AD+DB-2DB=2CB-2DB=2CD$

3 With the given line AB make the angle $ABC=\frac{1}{2}$ a right angle, with A as centre and the second line as radius describe a circle cutting BC at D and D', &c

4 See Ex 1 and take the several parts in pairs

5 Join the middle point of the hypotenuse with the right angle

Prop 10

1 Let AB be the given line, bisect it at the point D. Draw DE at right angles to AB. From the centre A and at the distance AB describe a circle cutting DE at E. Produce AB to C making DC equal to DE. The square on DC is three times the square on AD or DB

2 From the middle point C of AB draw CD at right angles to AB. With B as centre and BA as radius describe a ⊙ cutting CD at D. Produce AB to E making $BE=CD$

3 If the second line = the first, the produced part is nothing. When the second line is greater than the first —

Draw AF at right angles to AB the given line, and bisect AB at D, make $AE=AD$. Join ED and draw $DF \perp ED$. With E as centre and the second line as radius describe a ⊙ cutting DF at F. Draw $FC \perp AB$ produced

4 Proceed as in Ex 3, Prop 9. AB is the sum of the sides

5 Let D be the middle point of BC the base of the triangle ABC. Draw $AE \perp BC$.

$$\begin{aligned} AB^2+AC^2 &= 2AE^2+BE^2+EC^2 \\ &= 2AE^2+2BD^2+2DE^2 \\ &= 2BD^2+2AD^2 \end{aligned}$$

Prop. 11.

2 Apply Euc II 4, and II 5 *Cor*

3 Apply Euc II 3 and 2

4 Divide the given straight line AB at the point C, so that the rectangle contained by AB, BC may be equal to the square on AC (II 11) Produce AB to D making BD equal to BC Apply Euc II 10 and 7

5 $AH^2-HB^2=AB.BH-BH^2$
$$=(AB-BH)\times BH$$
$$=AH.BH$$

6 The angle FBA is equal to the angle ACH (I 4), &c

7 $AB^2+BH^2=2AB.BH+AH^2=3AH^2$ (II 7)

8 Produce FG, DB to meet in L, and join LC, LC passes through H, for $FH=HD$ $LH=LM=DN$

9 Let BD be $\perp$ AC, so that $AC.CD=AD^2$

$AC.CD=BC^2$ (Addl Prop II *Cor* page 186),

$\therefore$ $AD=BC$ Also $AB^2=AC.AD$ (Addl Prop II *Cor* p 186).
$$=AC.BC$$

Prop 12

1 Let ABC be the isosceles triangle, A being the vertex Make the angle ABD equal to the angle BAC BD cutting AC at D Draw BE perpendicular to AC AD, BD, BC are equal The square on AB is equal to the squares on BD, AD together with twice the rectangle contained by AD, DE, etc

2 The angle $BAD=\frac{1}{3}$ of a right angle
$$AC=BC=2CD,\ \&c$$

3 The angle $ACD=\frac{2}{3}$ of a right angle, and $AC=2CD$, &c.

5 See Addl Prop IV, page 188

6 Let ABC be the equilateral Δ, D a point in BC, is joined to A, $AE\perp BC$ Let D be in BE

$AB^2=AD^2+BD^2+2BD.DE$, and also$=BD^2+DC^2+2BD.DC$.

Again, $BD.DC=BE^2-DE^2$ (II 5)
$$=BD^2+DE^2+2BD.DE-DE^2 \text{ (II. 4)}$$
$$=BD^2+2BD.DE\text{, etc.}$$

Prop. 13.

1 Let BE be the perpendicular,

$AB^2 = AC^2 + BC^2 - 2BC.DC$

Again, $AB^2 = AC^2 + BC^2 - 2AC.EC$, $\therefore BC.DC = AC.EC$.

2 See Additional Prop III, page 187

3 Apply Ex 2

Prop 14

1 Let AB the side of the square, and BC equal to the given side, be so placed that ABC may become a right angle Join AC From A draw AD at right angles to AC meeting CB produced at D The rectangle contained by CB, BD is the required one. See Addl Prop II, page 186

2 Draw $BD \perp AB$ the given line

Bisect AB at C and make $BD = BC$ From CB produced cut off $CE = CD$ $\therefore$ $AE.EB = CE^2 - CB^2 = CD^2 - CB^2 = 2CB^2$, &c

Prop III

1 The diagonals AC, BD bisect each other at E

$AB^2 + BC^2 = 2AE^2 + 2BE^2$ $AD^2 + CD^2 = 2AE^2 + 2ED^2$

$\therefore AB^2 + BC^2 + AD^2 + CD^2 = 4AE^2 + 4BE^2 = AC^2 + BD^2$.

2 Let $ABCD$ be the quadrilateral, E the middle point of BD and F the middle point of AC

$AB^2 + BC^2 + CD^2 + DA^2 = 2AE^2 + 2BE^2 + 2AE^2 + 2ED^2$

$= 4AF^2 + 4BE^2 + 4EF^2$

$= AC^2 + BD^2 + 4EF^2$

Miscellaneous Exercises on Book II

1 Apply Euc II 4 and 1.

2 Let AB be the given straight line Suppose AC, CB be the two parts From CB cut off CD equal to AC We can prove by applying Euc II 6 and 8 that the square on AD is twice the square on DB Hence the construction At the point A in BA make the angle BAE equal to one-fourth of a right angle, and from B draw BF at right angles to AB At the point E in AE make the angle AED equal to the angle at A. Bisect AD at C AC and CB are the two parts

3. Draw BOD. The triangle ABD is double of the figure $ABED$, also equal to half the square on AB, therefore equal to the square on AO, &c.

4. Let ABC be the triangle, right-angled at B. The square on the sum of the sides AB, BC is equal to the squares on AB, BC together with twice the rectangle contained by AB, BC (II. 4).

5. Apply Euc. II. 5.

6. From the vertex A draw AD perpendicular to the base. Produce BC to E making CE equal to the difference of BD, DC.

7. See Addl. Prop. II, page 186, and also its *Cor.*

8. Let AB be the given straight line. From A draw AC at right angles to AB making it equal to the side of the given square. Through C draw CD parallel to AB. Bisect AB at E. With centre E and radius EA or EB, describe a circle cutting CD at F. Draw FG parallel to AC, meeting AB at G. Join AF, FB, and FE. AFB may be proved to be a right angle. Apply Ex. 7.

9. Let AB be the given straight line. From B draw BD at right angles to AB. From the centre A and with radius equal to the side of the given square, describe a circle cutting BD at D. Join AD. Draw DC at right angles to AD, meeting AB produced at C. Apply Ex. 7.

10. On the given straight line AB describe the triangle ACB so that AC, CB shall be equal to the two given straight lines. Draw CD perpendicular to AB. The difference of the squares on AC, CB is equal to the difference of the squares on AD, DB, which again is equal to the rectangle contained by AB and the difference of AD, DB.

11. Place AB, equal to the side of the given square, at right angles to BC, the given side of the rectangle. Draw AD at right angles to AC meeting CB produced at D. Apply Ex. 7.

12. Let AB be the given line. At A make the $\angle BAC = \frac{1}{4}$ rt. $\angle$, and at B make the $\angle ABC = \frac{1}{4}$ rt. $\angle$. Draw $CD \perp CA$, meeting AB at D. Then $AD^2 = 2BD^2$.

13. Let AB be the given straight line. From B draw BD at right angles to AB, making BD equal to a side of the given square. Bisect AB at C. Join CD. From CB produced, cut off CE equal to CD. Apply Euc. II. 6.

14. Produce the given line AB to D, making BD equal to AB. Produce again BD to C so that the rectangle contained by BC, CD shall be equal to the square on BD or AB [preceding problem]. We can prove by applying Euc. II. 2 and 3 that the square on BC is equal to the rectangle contained by AC, AB.

15 Let ABC be the triangle whose angle at B is two-thirds of a right angle From A draw AD perpendicular to BC. BD is half of AB [Ex 5, Prop 32, Book I.] Apply Euc II 13.

16 Apply Euc II 5 17 Apply Euc. II 9 and 5

18 Let AB be the diameter and PQD be the perpendicular, also let C be the centre Apply Ex 3, Prop 6, and prove that the square on AQ is equal to the rectangle AB, AD, and that the square on AP is equal to the rectangle AC, AD

19 Let AB be the given straight line At the point A in BA make the angle BAE equal to two thirds of a right angle, at the point B in AB make the angle ABE equal to half a right angle From E draw ED perpendicular to AB From DB cut off DC equal to AD AC, CB are the two parts Apply Euc II 10

20 Apply Euc II 5, *Cor*, and II 3

21 Let AB be the base On AB describe the rectangle AE equal to the difference of the squares on the sides that is equal to the rectangle contained by their sum and difference From BA cut off BC equal to BE the other side of the rectangle Bisect AC at D From D draw DF at right angles to AB DF is the required locus

22 The parts are the same as in Ex 19 Apply Euc II 7

23 Apply Euc II 13 or 7

24 Proceed as in Ex 15 and apply Euc II 12

25 From the points D, E draw perpendiculars on FB and KC produced &c

26 The angle FBA is equal to the angle ACH (I 4) &c

27 The vertical angle is equal to two thirds of two right angles.

28 Apply Euc II 12 and 13

29 to 32 Apply Addl Prop III, page 187

33 $AC.DB+CD.AB = AC.DB+AD.CD+CD.DB$

$$= AD.CD + AD.DB$$

$$= AD.CB$$

34 Apply Euc II, 13

35 Draw a perpendicular AE to the base and apply Euc II. 12 and 13

36 and 37. Apply Addl. Prop III page 187.

38. Let AB be the given difference From B draw BC at right angles to AB, and make it equal to the side of the given square. Bisect AB at D, join DC With D as centre and DC as radius, describe a circle cutting AB produced at E and F EB, BF are the sides of the required rectangle Euc II 5

39 Prove that the figure formed by joining the middle points of the sides is a parallelogram, any two adjacent sides of which are parallel to the diagonals and also each side is half of the diagonal to which it is parallel Apply Addl Prop III page 187

40 Let AB be the given straight line At the points A, B in AB make the angles BAE, ABE each equal to one fourth of a right angle At the point E in AE, BE, make the angles AEC, BED equal to the angle at A, B, respectively, let EC, ED meet AB at the points C, D

41 Draw the diagonals and apply Addl Prop III, page 187

42 Let AB be the given segment, at B draw BC at right angles to AB With A as centre and with radius equal to the given side, describe a circle cutting BC at C From C draw CD at right angles to AC, meeting AB produced at D

43 Describe a square equal to the given area (II 14) Let AB be the given difference Produce AB to C, so that the rectangle contained by AC, CB may be equal to the square described (Ex 13) From C draw CD at right angles to AC, making it equal to BC

44 Place the given segments AC, CB in a straight line Draw CD at right angles to AB Bisect AB at E With E as centre and EA as radius describe a circle cutting CD at D Join DA, DB ADB is the required triangle

45 Because the base and area are given, the altitude is known, and therefore the distance of the straight line parallel to the base and passing through the vertex of the triangle, is also known

46 By Ex 5, Book I 32 AE is double of BE, therefore the square on AE is four times the square on BE, hence the square on AB is three times the square on BE Similarly the square on BE is three times the square on EF or GB, etc

47 Divide the straight line, so that the rectangle contained by the whole line and one of the parts may be equal to the square on the other part (Euc II 11) Apply Euc II 7

48 Let AB be the given base, bisect it at C From the sum of the squares on the sides, take twice the square on AC or CB; the remainder shall be equal to twice the square on the line joining C with the vertex (Addl Prop III, page 187). Hence the line joining the vertex with the middle of the base can be found, and if with this line as radius and C as centre a circle be described, the circumference of the circle is the required locus.

49 Let AB be the given hypotenuse Divide AB into five equal parts (Ex. 4, Book I, p. 111), and let AC be one of the parts. With centres A, B, and radii equal to four times and three times AC respectively, describe two circles cutting each other at D. Join AD, BD ABD is the required triangle

50 Apply Euc. II 4, 1

51 Let ABC be any triangle, $ABDE$, $ACGF$, $BCHK$, the squares on the sides, EF, DK, HG the lines joining the angles of the squares Draw LM, KN, GP perpendiculars respectively on FA, DB, HC, produced wherever necessary Draw AQ, BR, CS perpendiculars from the angular points of the triangle on the opposite sides AM may be proved equal to AR, CP to CQ, and BN to BS (Euc I 26) Apply Euc II 12 and 13

52 See Ex 48, or apply Prop III

53 Let AB be the given straight line At the points A, B in AB, make the angles BAF, ABF, each equal to half a right angle From AB cut off AD equal to AE or BE Produce AB to C, making DC equal to AD Apply Euc II 7

54 Let AB be the given straight line Produce AB to E, and from BE cut off BC equal to AB From B draw BD at right angles to AB and equal to AB or BC Join DC From CE cut off CF equal to DC The square on CF is equal to the square on DC and therefore equal to twice the square on BC or AB Apply Euc II 10

Otherwise —From B draw BD at right angles to AB At A in BA make the angle BAD equal to three fourths of a right angle. At D in BD make the angle BDC equal to half a right angle, let AB produced meet DC at C

55. Join DB and apply Euc II 13

BOOK III.

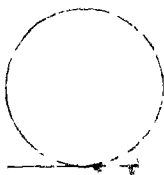
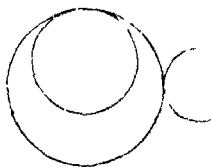
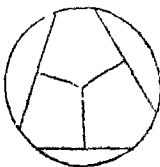
DEFINITIONS.

1. **Equal circles** are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal

2. A **chord** of a circle is the straight line joining any two points on the circumference of a circle

Each of the two parts into which a chord divides the circumference of a circle is called an **arc** —the greater the **major conjugate arc,** and the smaller the **minor conjugate arc.**

3. A straight line is said to **touch** a circle when it meets the circumference of the circle, and being produced does not cut it at any other point

The point at which the straight line touches the circle is called the **point of contact**, and the straight line which touches the circle is called a **tangent** to the circle.

4. Circles are said to **touch** one another, which meet, but do not cut one another

5. Straight lines are said to **be equally distant** from the centre of a circle, when the perpendiculars drawn to them from the centre are equal

6. And the straight line on which the greater perpendicular falls, is said to be further from the centre.

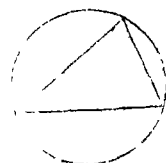
7 A **segment** of a circle is the figure contained by a chord and either of the two arcs into which the chord divides the circumference

Obs Every chord divides a circle into two segments

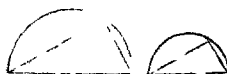
Obs The chord of a segment is sometimes called its *base*

8 The angle **of** a segment is that which is contained by the chord, and the circumference

Obs This angle is at the point where the chord and the arc meet, hence it is formed by a straight line and a curve It will however be shewn later on that the direction of the arc at any point is the same as that of the tangent at the same point In consequence we may define the angle of a segment as the angle formed by the chord and the tangent to the arc at the extremity of the chord

9 An angle **in** a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment

10 An angle is said to **insist** or **stand** on the arc intercepted between the straight lines which contain the angle

11 A **sector** of a circle is the figure contained by two straight lines drawn from the centre and the arc between them

12 **Similar segments** of circles are those in which the angles are equal, or which contain equal angles

13 Circles which have the same centre are called **concentric circles**

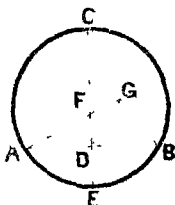
14 **Concyclic points** are those which lie on the circumference of the same circle.

15 A **cyclic rectilineal figure** is one whose angular points lie on the circumference of a circle.

Postulate A point, whose distance from the centre of a circle is less than the radius of the circle, is **within** the circle; and a point, whose distance from the centre of a circle is greater than the radius of the circle, is **without** the circle.

Proposition 1. Problem.

To find the centre of a given circle

Let ABC be the given circle: it is required to find its centre

Draw within it any straight line AB to meet the Oce in A, B,

and bisect AB at D, [I 10

from D draw DC ⊥ to AB meeting the Oce at C [I 11.

produce CD to meet the Oce at E,

and bisect CE at F [I 10

Then the point F shall be the centre of the circle ABC.

Now, the centre must lie in EC

for if not, let the centre be at G a point without EC

Join GA GB, GD

Then, in the two Δs ADG, BDG,

∵ AD = DB, [*Constr.*

the side GD is common,

and the radius AG = the radius GB [I *Def* 15.

∴ the ∠ ADG = the ∠ BDG, [I 8

∴ each of these angles is a right angle [I *Def.* 11

But the ∠ ADF = a right angle [*Constr.*

∴ the ∠ ADF = the ∠ ADG [*Ax* 11.

the part equal to the whole, which is absurd

∴ G is not the centre.

In a similar manner it may be proved that no point outside EC can be the centre.

∴ the centre lies in EC.

∴ F, the middle point of EC, must be the centre of the ⊙ ABC, since any other point in EC would divide EC unequally. Q. E. F.

Alternative Proof.

Because CE bisects AB at right angles,

∴ every point in CE is equally distant from A, B [I Prop XII.

But A, B, are equally distant from the centre

∴ the centre is in CE

Also, because the centre must be equally distant from C, E, the middle point of CE must be the centre Q E F

COR From this Proposition it is manifest that, in a circle, the straight line which bisects a chord at right angles passes through the centre.

EXERCISES.

1 Given the arc of a circle to find the centre of the arc, that is, of the circle of which it is an arc

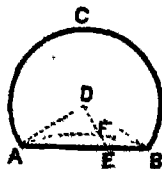
2 To describe a circle that shall pass through any three points which do not all lie in the same straight line

3 A number of circles pass through two fixed points, the locus of their centres is the line which bisects at right angles the line joining the fixed points

4 A, B, C, are three points on the circumference of a circle, the lines bisecting AB, BC, CA, perpendicularly are concurrent

Proposition 2 Theorem

If any two points be taken on the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle,

and A, B, any two points in the circumference ·

then the straight line drawn from A to B shall fall within the circle.

For if it do not, let it fall, if possible, without, as AEB.

Find D the centre of the circle ABC ; [III. 1.

and join DA, DB;
in the arc AB take any point F;
join DF, and produce it to meet AB at E.
Then in the △ DAB,

∵ DA = DB, [I. *Def.* 15.

∴ the ∠ DAB = the ∠ DBA. [I. 5.

But in the △ DAE, the ext ∠ DEB is greater than the int. and opp. ∠ DAE, [I. 16.

∴ the ∠ DEB is greater than the ∠ DBE.

∴ in the △ DEB,

the side DB is greater than the side DE. [I 19.

But the radius DB = the radius DF, [I. *Def* 15.

∴ DF is greater than DE:

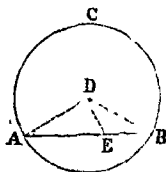
the part greater than the whole, which is absurd.

Therefore the straight line drawn from A to B does not fall without the circle

In the same manner it may be demonstrated that it does not fall on the ○ce.

Therefore it falls within the circle.

Wherefore, *if any two points, &c* Q. E. D.

Alternative Proof

Find D the centre of the ⊙ ACB; join AD, BD.
Take any point E in AB, join DE

The ∠ DEA is greater than the ∠ DBE [I. 16.

But the ∠ DBA = the ∠ DAB, [I. 5.

∴ the ∠ DEA is greater than the ∠ DAE

∴ the side AD is greater than the side DE [I. 19.

∴ DE is less than AD the radius of the circle.

∴ the point E falls within the ○ce of the circle ACB [*Post.*

Likewise, we can shew that every point in AB falls within the ○ce of the circle.

Wherefore, AB falls within the ○ce of the circle. Q. E. D.

Def. A part of a curve is ***concave*** to any fixed point P, when any chord RS being taken, no straight line joining P and any point in RS is cut by the arc intercepted by RS, and a part of a curve is ***convex*** to P when all straight lines joining P and any point in RS is cut by the arc intercepted by RS.

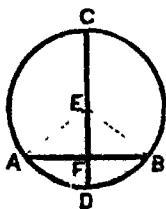
EXERCISES.

1. The circumference of a circle is everywhere concave towards its centre

2. A circle cannot pass through three collinear points

Proposition 3 Theorem.

If a straight line, drawn through the centre of a circle, bisect a chord which does not pass through the centre, it shall cut it at right angles, and conversely, if it cut it at right angles it shall bisect it

Let ABC be a circle,
and let CD, a straight line drawn through the centre, bisect any chord AB, which does not pass through the centre, at the point F.
then CD *shall cut* AB *at right angles.*

Find E the centre of the ⊙, and join EA, EB. [III. 1.
Then, in the two △s AFE, BFE,

AF = FB, [*Hyp.*
EF is common,
and the radius EA = the radius EB:
∴ the ∠ AFE = the ∠ BFE. [I. 8.
∴ each of these angles is a right angle. [I. *Def.* 11.
∴ CD cuts AB at right angles

CONVERSELY:—let CD cut AB at right angles.
then CD *shall bisect* AB.

Make the same construction as before.

Then, ∵ EA = EB,

∴ the ∠ EAB = the ∠ EBA. [I. 5.

Hence, in the two △s AFE, BFE,

the ∠ EAF = the ∠ EBF,

the ∠ AFE = the ∠ BFE, being right angles,

and EF is common.

∴ AF = FB [I. 26.

∴ CD bisects AB.

Wherefore, *if a straight line*, &c. Q. E. D.

Alternative Proof of the Second Part.

The sq. on AE = the sq. on EB.

∴ the sqs. on AF, FE = sqs. on BF, FE. [I. 47.

Take away the common sq. on FE.

Then the sq. on AF = the sq. on FB.

∴ AF = FB. [I. 46, *Cor.* 3.

EXERCISES.

1. Two circles cut each other; the straight line joining their centres bisects their common chord at right angles.

2. *If a straight line cut the circumference of two concentric circles, the segments of the line intercepted between the circles are equal.*

3. Through a given point within a circle, which is not the centre, draw a chord which shall be bisected at that point.

4. The locus of the points of bisection of all parallel chords in a circle is a diameter at right angles to all of them.

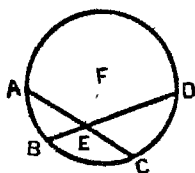
5. To describe a circle which shall pass through two fixed points and shall have its centre in a given straight line.

6. In a circle, a chord of given length subtends a right angle at a given point; find the locus of the middle point of the chord.

Proposition 4 Theorem.

If in a circle two chords cut one another, which do not both pass through the centre, they do not bisect one another.

Let ABCD be a circle, and AC, BD, two chords in it, which cut one another at the point E, and do not both pass through the centre :

then AC, BD, *shall not bisect one another.*

First If *one* of the chords passes through the centre, it is a diameter, and therefore cannot be bisected by the other chord which does not pass through the centre.

Secondly If *neither* of the chords passes through the centre :—

then, if possible, let AE = EC, and BE = ED ;

Find F, the centre of the circle, [III. 1.

and join FE

Then, since FE, drawn through the centre, bisects AC which does not pass through the centre,

∴ the ∠ AEF is a right angle. [III 3.

Similarly, since FE, drawn through the centre, bisects BD which does not pass through the centre,

∴ the ∠ BEF is a right angle.

∴ the ∠ AEF = the ∠ BEF ·

the part equal to the whole, which is absurd.

Therefore AC, BD do not bisect one another.

Wherefore, *if in a circle*, &c. Q. E. D.

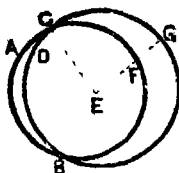
EXERCISE.

If in a circle, two chords bisect one another they are both diameters.

Proposition 5. Theorem.

If two circles cut one another they are not concentric.

Let the two circles ABC, CDG cut one another at the points B, C.

then they are not concentric, that is, they cannot have the same centre

For, if it be possible, let E be their common centre:

join EC, *and draw any straight line* EFG *meeting the* ⊙ces at F and G

Then, since E is the centre of the circle ABC,

∴ the radius EC = the radius EF.

Similarly, since E is the centre of the circle CDG,

∴ the radius EC = the radius EG.

∴ EF = EG:

the part equal to the whole, which is impossible.

Therefore E is not the common centre of the circles ABC, CDG.

Wherefore, *if two circles*, &c. Q. E. D.

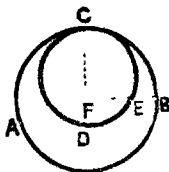
EXERCISE

One circle cannot meet another in three points without wholly coinciding with it.

Proposition 6 Theorem.

If one circle touch another internally, they are not concentric.

Let the circle CDE touch the circle ABC internally at the point C;

then they are not concentric.

For, if it be possible, let F be their common centre

join FC, and draw any straight line FEB meeting the Oces at E and B

Then, since F is the centre of the circle CDE,

∴ the radius FC = the radius FE

Similarly, since F is the centre of the circle ABC,

∴ the radius FC = the radius FB

∴ FE = FB

the part equal to the whole which is impossible

Therefore F is not the common centre of the circles ABC, CDE.

Wherefore, *if one circle, &c* Q E D.

EXERCISES

1. One circle cannot touch another internally in two points without wholly coinciding with it

2 Two unequal concentric circles cannot have any point in common

Proposition 7. Theorem.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of the diameter is the least;

and of the rest, that which is nearer to the straight line which passes through the centre, is always greater than one more remote;

and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the diameter

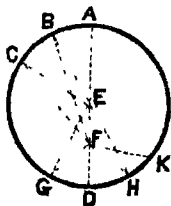
Let ADCB be a circle and AD its diameter, in which let any point F be taken which is not the centre; let E be the centre Let FA, FB, FC, FG, FD, etc, be drawn to the ○ce, of which FA passes through the centre E, and FD is the other part of the diameter, also, of the rest, let FB be nearer to FA than is FC, and FC nearer than FG

then of all these straight lines,

(*a*) FA *shall be the greatest,*

(*b*) FD *shall be the least,*

(*c*) FB *shall be greater than* FC, *and* FC *greater than* FG,

(*d*) *finally, two and not more than two equal straight lines can be drawn from* F *to the* ○ce, *one on either side of the diameter* AFD

Join EB, EC, and EG.

(*a*) Then in the △ BEF,

FE and EB are together greater than FB, [I. 20.

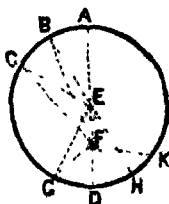
but the radius EB = the radius EA.

∴ FE and EA are together greater than FB,

that is, FA is greater than FB

In a similar manner it may be proved that FA is greater than any other straight line drawn from F to the ○ce.

∴ FA is the greatest straight line that can be drawn from F to the ○ce,

(b) In the △ EFG,
EG is less than the sum of EF and FG; [I. 20.
but the radius EG = the radius ED:
∴ ED is less than the sum of EF and FG;
∴ taking away the common part EF,
FD is less than FG.

In a like manner it may be shewn that FD is less than any other straight line drawn from F to the ○ce.
∴ of all such lines FD is the least.

(c) In the △s FEB, FEC,
EB = EC, being radii of the circle,
FE is common,
but the ∠ FEB is greater than the ∠ FEC:
∴ FB is greater than FC [I. 24.
Likewise it may be proved that FC is greater than FG.

(d) At E in FE,
make the ∠ FEH = the ∠ FEG, [I. 23.
and join FH.
Then in the △s FEG, FEH,
the radius EG = the radius EH,
EF is common,
and the ∠ FEG = the ∠ FEH:
∴ FG = FH. [I. 4.

And no other straight line can be drawn from F to the ○ce equal to FG.
For, if possible, let FK = FG.
Then FK = FG = FH:
that is FK, being nearer to FA the greatest line, is equal to FH which is more remote; which is absurd.

Wherefore, *if any point be taken*, &c. Q. E. D.

EXERCISES

1. From a point within a circle draw the longest line to the circumference, also draw the shortest.

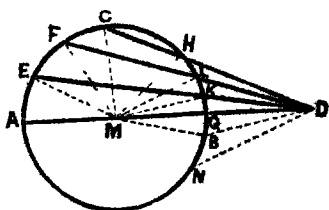
2 From a point within a circle draw the diameter without finding the centre.

Proposition 8 Theorem.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre :

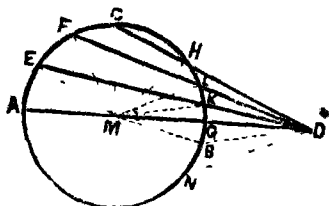
of those which fall on the concave part of the circumference the greatest is that which passes through the centre , and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote ,

but of those which fall on the convex part of the circumference, the least is that which when produced passes through the centre ; and of the rest, that which is nearer to the least is always less than one more remote ;

and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the line joining the centre with the given point

Let ABC be a circle, and D any point without it, and from D, let the straight lines DA, DE, DF, DC, DH, DL, DK, etc. be drawn to the circumference, of which DA passes through the centre :

then of those which fall on the concave part of the circumference AEFC, *the greatest shall be* DA *which passes through the centre, and any line nearer to it shall be greater than one more remote, namely* DE *shall be greater than* DF, *and* DF *greater than* DC ;

but of those which fall on the convex part of the circumference GKLH, *the least shall be* DG *which when produced passes through the centre* M, *and the nearer to the line* DG *shall be less than the more remote, namely* DK *less than* DL, *and* DL *less than* DH,

finally, not more than two equal straight lines can be drawn from D *to the circumference*

Take M, the centre of the circle ABC, [III. 1.
and join ME, MF, MC, MH, ML, MK.

Then, in the $\triangle$ DME

the sum of DM and ME is greater than DE; [I. 20.

but the radius ME = the radius MA

$\therefore$ the sum of DM and MA is greater than DE,

that is, DA is greater than DE

Also, in the $\triangle$s DME, DMF,

DM and ME are respectively equal to DM and MF,

but the $\angle$ DME is greater than the $\angle$ DMF

$\therefore$ DE is greater than DF [I. 24.

Similarly, DF is greater than DC

$\therefore$ of the lines DA, DE, DF, and DC,

DA is the greatest

moreover DE is greater than DF, and DF greater than DC.

Again, in the $\triangle$ DKM,

DM is less than the sum of DK and KM, [I. 20.

that is, the sum of DG and GM is less than the sum of DK and KM,

but the radius GM = the radius KM.

$\therefore$ DG is less than DK

Also, in the $\triangle$s DKM, DLM,

the sum of DK and MK is less than the sum of DL and ML;
[I. 21.

but MK = ML

∴ DK is less than DL

Similarly, DL is less than DH

∴ of the straight lines DG, DK, DL, DH,

DG is the least,

and DK less than DL, DL less than DH.

Finally, at M make the ∠ DMB = the ∠ DMK, [I 23.

and join DB

Then in the △s DMB, DMK,

MB = MK, being radii,

DM is common,

and the ∠ DMB = the ∠ DMK,

∴ DB = DK [I 4.

And no other straight line from D can be drawn to the ○ce equal to DK

For, if possible, let DN = DK.

∴ DN = DK = DB

that is, DB which is nearer to DG is equal to DN which is more remote, which is impossible [*Proved*

∴ only two equal straight lines can be drawn from D to the ○ce, one on each side of DM

Wherefore, *if any point be taken, &c.* Q. E. D.

Alternative Proof.

This Proposition may be enunciated and proved thus —

Of all straight lines which can be drawn from a point without a circle to any point on the circumference, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre, and of the rest, the one which subtends a greater angle at the centre is greater than one which subtends a less angle, and from the same point not more than two equal straight lines can be drawn to the circumference, one on each side of the line joining the centre with the given point

Let G, K, L, H, C, F, E, A, be points taken on the ○ce, and let DGA pass through the centre M.

Then, in each of the △s formed,

the line joining D with one of the above points (G, K, L, H, etc) may be considered to be the base,

and, the fixed line DM and the radius, to be the sides.

Then in the series of △s,

the two sides would be identically equal,

for, DM would be common, and the radius from M to each point on the ○ce (G, K, L, H, etc.) would be all equal;

but the vertical ∠ subtended at the centre M by each of the bases DG, DK, DL, DH, DC, etc, would be greater and greater:

∴ by I 24, in the series of △s the bases DG, DK, DL, DH, DC, DF, DE, DA, would be greater and greater

But when the point G on the ○ce is such that DG produced passes through the centre M, the vertical angle subtended by the base DG at the centre M is the least possible in a triangle, for it is nothing,

and when the point A on the ○ce is such that DA passes through the centre M, the vertical angle subtended by the base DA at the centre M is the greatest possible in a triangle, for it is equal to two right angles

∴ DG is the least,

and DA is the greatest.

Finally, at the point M in DM only two angles can be made equal to one another, one on each side of MD,

∴ only two △s like the above series can have their bases equal, one on each side of MD,

∴ only two equal straight lines can be drawn from D to the ○ce, one on each side of DM Q. E. D.

NOTE. Prop. 7 may be enunciated and proved in exactly the same manner The student might try to do so as an exercise

EXERCISES

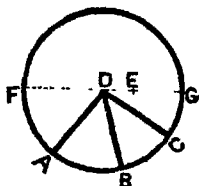
1 From a point without a circle draw the shortest straight line to the convex circumference, also the longest line to the concave circumference

2. From a point outside a circle draw the diameter without finding the centre.

Proposition 9. Theorem.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, namely, DA, DB, DC

then the point D *shall be the centre of the circle.*

For, if not, let E be the centre;

join DE and produce it both ways to meet the ○ce at F and G, then FG is a diameter of the circle ABC.

Then, because D is a point in FG the diameter, which is not the centre,

∴ DC, which is nearer to DG, is greater than DB, which is more remote, and DB is greater than DA · [III 7.

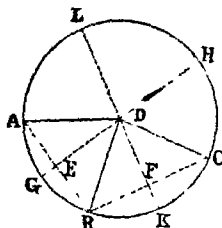
which is absurd, since by hypothesis all these lines are equal.

∴ E is not the centre of the circle

Similarly it may be proved that if any point other than D be taken, it cannot be the centre, since it would lead to the absurdity that at least two of the given straight lines DA, DB, DC, are unequal.

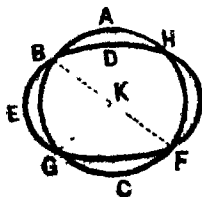
Wherefore, *if a point be taken, &c.* Q. E. D.

Alternative Proof.

Join AB, BC, and bisect them at E and F, respectively. Join ED, DF, and produce them to meet the ⊙ at G, H, and L, K.

∵ AD=DB, AE=EB, and ED is common ;
∴ the ∠ AED=the ∠ BED, [I. 8.
∴ each of them is a right angle
∴ GH bisects AB at right angles
∴ the centre of the circle is in GH [III 1. *Cor*
Likewise, we can prove that LK bisects BC at right angles, and the centre of the circle is in LK [III. 1. *Cor.*
But D is the only point common to GH and LK.
∴ D is the centre of the circle Q E D

Proposition 10 Theorem

One circumference of a circle cannot cut another at more than two points

If it be possible, let the circumference ABC cut the circumference DEF at more than two points, namely, at the points B, G, F

Take K, the centre of the ⊙ ABC, [III. 1.
and join KB, KG, KF.

Then, since K is the centre of the ⊙ ABC,

∴ KB=KG=KF.

And because within the ⊙ DEF, K is a point such that more than two equal straight lines are drawn from it to the ○ce namely, KB, KG, KF:

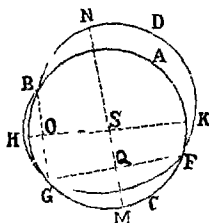
∴ K is the centre of the ⊙ DEF. [III. 9.

But K is also the centre of the circle ABC [Cons.

Therefore the same point is the centre of two circles which cut one another

which is impossible [III. 5.

Wherefore, *one circumference*, &c Q. E. D.

Alternative Proof.

Join BG, GF, and bisect them at O and Q;

from O draw HOK at right angles to BG,

and through Q draw NQM at right angles to GF.

Let NM cut HK at S

Since HK bisects BG at right angles,

∴ the centre of each ⊙ is in HK. [III. 1. *Cor.*

Also, since NM bisects GF at right angles,

∴ the centre of each ⊙ is in NM: [III. 1. *Cor.*

∴ the centre of each ⊙ is S where NM and HK intersect; which is impossible [III. 5.

Wherefore, *one circumference, &c.* Q. E. D.

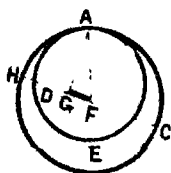
Proposition 11. Theorem.

If one circle touch another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the circle ADE, touch the circle AHC internally at the point A. and let F be the centre of the circle AHC, and G the centre of the circle ADE

the straight line which joins the centres F, G, *being produced, shall pass through the point* A

For, if not, let it pass otherwise, if possible, as FGDH, and join AF, AG

Then, since AG, GF are greater than FA, [I 20.

and FA = FH, being radii of the ⊙ AHC.

∴ AG, GF are greater than FH

Take away the common part GF,

∴ GA is greater than GH

But GA = GD, being radii of the ⊙ ADE.

∴ GD is greater than GH

the part greater than the whole, which is absurd.

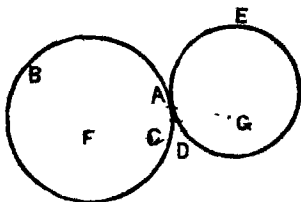
Therefore the straight line which joins the points F, G, being produced, cannot pass otherwise than through the point A, that is, it must pass through A

Wherefore, *if one circle, &c.* Q. E. D.

Proposition 12. Theorem.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch one another externally at the point A and let F be the centre of the circle ABC, and G, the centre of the circle ADE :

then the straight line which joins the points F, G, *shall pass through the point* A.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG.

Then, since F is the centre of the ⊙ ABC,

∴ FA = FC, being radii.

Also, since G is the centre of the ⊙ ADE,

∴ GA = GD, being radii.

∴ the sum of FC, GD = the sum of FA, GA.

∴ the whole FG is greater than the sum of FA, GA.

But FG is also less than the sum of FA, GA. [I. 20.

which is absurd.

Therefore the straight line which joins the points F, G, cannot pass otherwise than through the point A,

that is, FG must pass through A.

Wherefore, *if two circles*, &c. Q. E. D.

EXERCISES

1. If two circles touch one another, the straight line joining the centre of one of the circles with the point of contact, produced when the circles touch externally, shall contain the centre of the other circle.

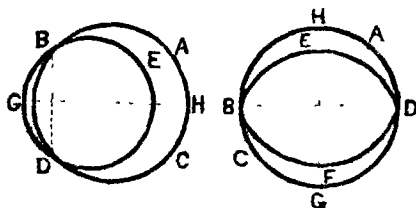
2 If two circles touch each other externally, and a straight line be drawn through the point of contact, cutting them both, the diameters drawn through the points of section shall be parallel

Proposition 13. Theorem.

One circle cannot touch another at more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle EBD touch the circle ABC at more points than one

1st Case First on the inside, at the points B, D.

Join BD, and draw GH bisecting BD at right angles
[I 10, 11.

Then, since the two points B, D, are on the Oce of each circle,

the chord BD falls within each circle, [III 2.

the centre of each ⊙ lies in GH which bisects BD at right angles [III 1, Cor.

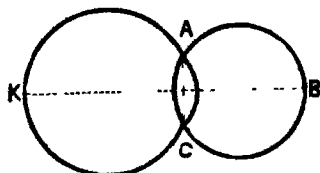
.. GH passes through the point of contact [III 11.

But GH does not pass through the point of contact, because the points B D are out of the line GH,

which is absurd

. one circle cannot touch another on the inside at more points than one.

2ND CASE **Nor can two circles touch one another on the outside at more points than one.**

For if it be possible, let the circle ACK touch the circle ABC at the points A, C. Join AC.

Then, since the points A, C, lie on the $\bigcirc^{ce}$ of the $\odot$ ACK,
∴ the chord AC falls within the $\odot$ ACK [III 2.
But the $\odot$ ACK is without the $\odot$ ACB, [*Hyp*
∴ the chord AC is without the $\odot$ ACB
But since the points A, C, lie on the $\bigcirc^{ce}$ of the $\odot$ ACB,
∴ the chord AC falls within the $\odot$ ACB [III 2.
which is absurd

Therefore one circle cannot touch another on the outside at more points than one

And it has been shewn that one circle cannot touch another on the inside at more points than one

Wherefore, *one circle cannot*, &c Q. E. D

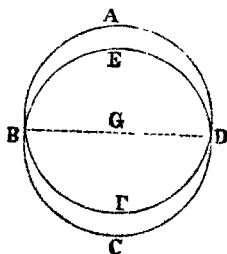
Alternative Proof.

1ST CASE When one circle touches another *internally*.

Since the $\odot$ EBF touches the $\odot$ ABC at B,
∴ the straight line joining their centres passes through the point B [III 11

Also, since the $\odot$ EBF touches the $\odot$ ABC at D,
∴ the straight line joining their centres passes through the point D [III 11

∴ the centres and B, D, are collinear

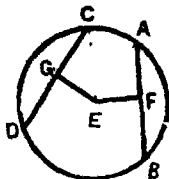
∴ BD is a diameter of each circle.
Bisect BD at G,
G is the centre of each circle.
∴ the circles are concentric, which is impossible. [III 6.

2ND CASE When the circles touch each other *externally*.

The straight line joining the centres of the circles must pass through each point of contact, which is impossible [III 12.

Proposition 14 Theorem

Equal chords in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another

Let the chords AB, CD, in the circle ABDC be equal to one another

then they shall be equally distant from the centre

Take E, the centre of the circle ABDC [III 1.
and from E draw EF, EG perp to AB, CD [I 12.
join EA, EC

Then, since EF passing through the centre is perp to the chord AB, which does not pass through the centre

∴ EF bisects AB, [III 3.

∴ AB is double of AF

Similarly, CD is double of CG

But AB = CD, [*Hyp.*

∴ AF = CG [*Ax* 7.

Now, EA = EC, being radii,

∴ the sq on EA = the sq on EC

But the sq on EA = the sqs on EF, AF, [I. 47.
for the ∠ F is a right angle

And the sq on EC = the sqs on EG, CG, [I 47.
for the ∠ G is a right angle

∴ the sqs. on EF, AF = the sqs on EG, CG

But the sq. on AF = the sq. on CG,
∴ the sq. on EF = the sq. on EG,
∴ EF = EG

that is, the chords AB, CD, are equally distant from the centre.

Conversely Let AB, CD, be equally distant from the centre E,

that is, let EF = EG

then shall AB = CD

With the same construction as before, it may be proved that

AB is double of AF,
and CD is double of CG,

moreover, that the sqs. on EF, AF = the sqs. on EG, CG [I. 47.

But the sq. on EF = the sq. on EG,
∴ the sq. on AF = the sq. on CG
AF = CG,
and their doubles are equal, [Ax. 6
∴ AB = CD

Wherefore, equal straight lines &c. Q. E. D.

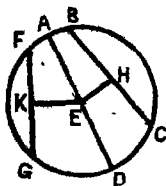
EXERCISES

1. In a circle two chords, which cut a diameter at the same point at equal angles, are equal

2. If any number of equal chords in a circle be bisected, one circle passes through all the points of bisection

Proposition 15. Theorem.

The diameter is the greatest chord in a circle; and of all others, that which is nearer to the centre is always greater than one more remote; and conversely, the greater chord is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG: *then* AD *shall be greater than any chord* BC *which is not a diameter; and* BC *shall be greater than* FG.

From the centre E draw EH, EK perp. to BC, FG, [I. 12.

and join EB, EC, EF.

Then, in the △ BEC,

the sum of EB, EC is greater than BC; [I. 20.

∴ the sum of two radii is greater than BC,

∴ a diameter is greater than BC,

that is, AD is greater than BC.

And since BC is nearer to the centre than FG, [*Hyp.*

∴ EH is less than EK. [III. *Def.* 6.

Now, it may be demonstrated, as in the preceding Proposition,

that BC is double of BH,

and FG double of FK,

and that the sqq. on EH, HB = the sqq. on EK, KF.

But the sq. on EH is less than the sq. on EK,

because EH is less than EK;

∴ the sq. on HB is greater than the sq. on KF,

∴ the straight line BH is greater than the straight line FK;

∴ BC is greater than FG.

Conversely. **Let BC be greater than FG;**

then BC *shall be nearer to the centre than* FG,

that is, the same construction being made,

EH *shall be less than* EK

For, because BC is greater than FG,

BH is greater than FK

But the sqs on BH, HE, = the sqs on FK, KE,

and the sq on BH is greater than the sq on FK,

because BH is greater than FK

∴ the sq on HE is less than the sq on KE,

∴ the straight line EH is less than the straight line EK:

∴ BC is nearer to the centre than FG

Wherefore, *the diameter*, &c Q E D

EXERCISES.

1 The shortest chord which can be drawn through a given point within a circle, is that which is perpendicular to the diameter passing through that point

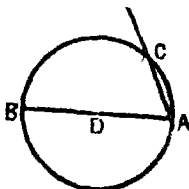
2 The locus of the middle points of chords drawn from a fixed point in the circumference of a circle, is a circle whose radius is half that of the given circle

3 Through a given point draw a chord equal to a given straight line not greater than the diameter of the circle

Proposition 16. Theorem.

The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle, and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.

Let ABC be a circle, of which D is the centre and AB a diameter:

Then the straight line drawn at right angles to AB, *from its extremity* A, *shall fall without the circle.*

For, if not, let it fall, if possible, within the ⊙, as AC, and draw DC to the point C, where it meets the ○ce

Then, in the △ ADC,

∵ DA = DC, being radii,

∴ the ∠ DAC = the ∠ DCA [I 5

But the ∠ DAC = a rt angle, [Hyp.

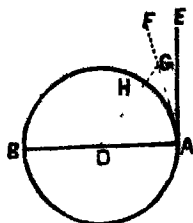
∴ the ∠ DCA = a rt angle

∴ two ∠s of the △ ADC together = two rt angles: which is impossible [I 17.

∴ the straight line drawn from A at right angles to AB does not fall within the ⊙.

And in the same manner it may be demonstrated that it does not fall on the ○ce

∴ it must fall without the ⊙, as AE.

Also, between AE and the ○ᶜᵉ, no straight line can be drawn from the point A, which does not cut the ⊙.

For, if possible, let AF be between them ,
and from the centre D draw DG perp to AF ; [I. 12.
let DG meet the ○ᶜᵉ at H.

Then, because the ∠ DGA is a right angle, [*Cons.*
∴ the ∠ DAG is less than a right angle , [I. 17.
∴ DA is greater than DG. [I. 19.
But DA = DH ; [I. *Def.* 15.
∴ DH is greater than DG .
the part greater than the whole , which is impossible.

∴ no straight line can be drawn from the point A, between AE and the ○ᶜᵉ, so as not to cut the circle.

Wherefore, *the straight line*, &c. Q. E. D.

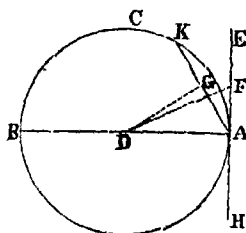
COR From this Proposition it is manifest that the straight line drawn at right angles to the diameter of a circle from the extremity of it, touches the circle , [III. *Def.* 2
and that it touches the circle at one point only,
because if it met the circle at two points, it would fall within it. [III. 2.

Also, it is evident that there can be but one straight line which touches the circle at the same point

We can prove this Proposition in the following manner :—

A straight line drawn at right angles to the diameter of a circle from an extremity of it touches the circle, and any other straight line from the same point cuts the circle

Let AE be at right angles to AB the diameter

Then AE *touches the circle*

Take any point F in AE, and join DF

Because the ∠ DAF is a right angle,

∴ the ∠ DAF is greater than the ∠ DFA [I 32.

∴ DF is greater than DA [I. 19.

∴ F is without the circle [*Post.*

Likewise we can prove, that every point in AE, except the point A, is without the circle

∴ AE is without the circle and meets it only at A

∴ AE touches the circle

If EA be produced to H, we can likewise prove that AH touches the circle

Wherefore EAH touches the circle

Again from A draw a straight line AK making an ∠ less than a right angle

Draw DG perp to AK

The ∠ DGA is greater than the ∠ DAG [I 32

∴ DA is greater than DG.

∴ G is within the circle [*Post.*

Likewise, we can prove that every point in AK is within the circle

∴ AK falls within the circle.

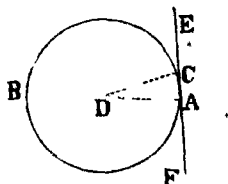
∴ AK cuts the circle. Q. E. D.

ANOTHER PROOF.

The tangent drawn from any point on the circumference of a circle is at right angles to the radius from that point

Let A, C, be two consecutive points on the ○ce.

Join CD, AD, AC

Produce AC both ways

Because A, C, are consecutive points on the ○ce,

∴ they are as if one point

∴ EF is a tangent [See *Notes on Book III Def* 3.

Also the sum of the three ∠s of the △ ADC = two right angles [I 32.

But the ∠ CDA is infinitely small or is nothing,

and the ∠s DCA DAC are equal,

each of DCA, DAC is a right angle,

and the straight lines DC, DA, coincide with one another

∴ EF is at right angles to AD Q E D

Otherwise

A tangent is the limiting position of a secant When the chord portion of the secant is so reduced that the extremities of the chord are two consecutive points, that is, when the secant becomes a tangent, the straight line through the centre which bisects the chord portion of the secant is perpendicular to the same portion (III 3), and therefore perpendicular to the limiting position of the secant or to the tangent Q E D

Otherwise

The ∠ DAF is always equal to the ∠ DCE,

∴ when A, C, come together, each of the ∠s DCE and DAF is a right angle

∴ the tangent is perp to the radius Q. E. D.

EXERCISES.

1 The tangents at the extremities of the same diameter are parallel

2. Draw a straight line which shall be perpendicular to a given straight line and shall touch a given circle

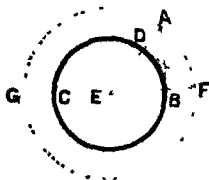
3 Describe a circle that shall touch two given circles

4. To describe a circle which shall have a given radius, and shall have its centre in a given straight line, and shall also touch another given straight line

5 A rod of given length slides within a hoop, so that its two ends always touch the hoop; shew that the rod is always a tangent to a fixed circle

Proposition 17 Problem

To draw a straight line from a given point either without or on the circumference, which shall touch a given circle

First, let the given point A be without the given circle BCD; it is required to draw from A a straight line, which shall touch the given circle

Take E, the centre of the circle, [III. 1.
and join AE cutting the $\bigcirc^{ce}$ of the given circle at D.

From the centre E, at the distance EA, describe the $\odot$ AFG,

from D draw DF at right angles to EA, [I. 11.
and join EF, cutting the $\bigcirc^{ce}$ of the given circle at B;
join AB.

AB *shall touch the circle* BCD.

For, in the $\triangle$s BEA, DEF,

BE = DE, being radii of the ⊙ BCD,

and EA = EF, being radii of the ⊙ AFG,

and the ∠ E is common to both ;

∴ the ∠ EBA = the ∠ EDF [I. 4.

= a right angle. [*Constr*.

∴ BA, being at right angles to the diameter through B, is a tangent [III 16 *Cor*.

And AB is drawn from the given point A Q E F

But if the given point be on the $\bigcirc^{ce}$ of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE, *then* DF *touches the circle* [III 16. *Cor*

COR 1 *It is evident that from an external point two tangents can be drawn to a circle* For, if FD be produced to meet the outer ⊙ at H and HE cut the inner ⊙ at K, then AK is another tangent

COR 2 These two tangents make equal angles with the line joining the given point A to the centre E

COR 3 These tangents subtend equal angles at the centre

OBS When the point A comes up to the $\bigcirc^{ce}$ the two tangents are in the same straight line, hence there would be only one tangent And when the point comes within the circle, no tangent can be drawn from it to the circle

EXERCISES

1 The two tangents drawn from a point without the circumference of a circle, are equal Shew that a third tangent cannot be drawn from the same point

2 If a quadrilateral ABCD be described about a circle, prove that the sum of AD, BC is equal to the sum of AB, CD

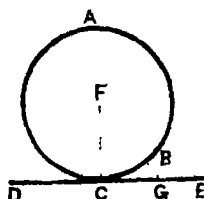
3 The parallelogram which can be circumscribed about a circle is a rhombus, and its diagonals intersect at the centre of the circle

4 *If two circles be concentric and if a chord of the greater circle cut the smaller, the intercepts between the two circles are equal.*

Proposition 18 Theorem.

The straight line drawn from the centre of a circle to the point of contact of a tangent is perpendicular to the tangent.

Let the straight line DE touch the circle ABC at the point C

take F, the centre of the circle ABC, and join FC :
then FC *shall be perpendicular to* DE

For, if not, let FG be drawn from F perp to DE, meeting the O^ce at B

Then, because FGC is a right angle [Hyp

FCG is an acute angle [I 17.

∴ FC is greater than FG [I 19

But FC = FB [I Def 15

∴ FB is greater than FG

the part greater than the whole which is impossible

∴ FG is not perp to DE

In the same manner it may be shewn that no other straight line from F, but FC is perp to DE

∴ FC is perp to DE

Wherefore, *the straight line*, &c Q E D

EXERCISES

1 In two concentric circles, if a chord of the greater touch the less, the chord is bisected at the point of contact.

2 If two circles be concentric, all chords of the greater circle which touch the smaller circle are equal

3 If two straight lines intersect, the loci of the centres of all circles touched by both lines are two straight lines at right angles to each other

4 Describe a circle passing through a given point, and touching a given straight line at a given point.

5 Describe a circle passing through a given point, and touching a given circle at a given point

6 Describe a circle touching a given circle, and touching a given straight line at a given point

7. Describe a circle touching a given circle at a given point, and touching a given straight line

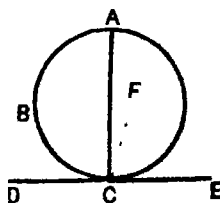
8 Describe a circle touching two given circles, the second circle in a given point

9 Describe a circle touching two given straight lines and a third not passing through the point of intersection of the first two.

10 Draw a common tangent to two circles

Proposition 19 Theorem.

If from the point of contact of a tangent to a circle a straight line be drawn at right angles to the tangent, it shall pass through the centre

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE, *then the centre of the circle shall be in CA*

For, if not, if possible let F be the centre, and join CF.
Then because DE touches the ⊙ ABC, and FC is drawn from the centre to the point of contact,

∴ FC is perp to DE, [III 18.

∴ the ∠ FCE is a right angle

But the ∠ ACE is also a right angle, [Constr.

∴ the ∠ FCE = the ∠ ACE [Ax 11.

the part equal to the whole which is impossible

∴ F is not the centre of the ⊙ ABC

In the same manner it may be shewn that no other point out of CA is the centre,

∴ the centre is in CA

Wherefore, *if from the point*, &c. Q. E. D.

EXERCISE.

Through a given point within or without a circle to draw a chord equal to a given straight line not greater than the diameter of the circle (If the point be within the circle, the given straight line should not be less than the chord drawn through the given point at right angles to the diameter passing through that point. (See Ex. 1, Prop 15)

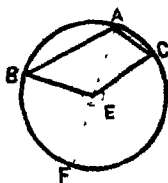
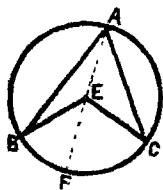
Proposition 20 Theorem.

The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC, for their base

then the angle BEC *shall be double of the angle* BAC.

Join AE and produce it to F

CASE I Let the centre of the ⊙ be within the ∠ BAC.

Then, since EA = EB,

∴ the ∠ EAB = the ∠ EBA [I 5.

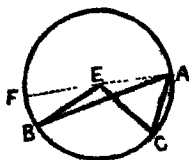
∴ the sum of the ∠s EAB, EBA = double the ∠ EAB.

But the sum of the ∠s EAB, EBA = the ∠ BEF, [I. 32.

∴ the ∠ BEF = double the ∠ EAB

Similarly, the ∠ FEC = double the ∠ EAC.

∴ the whole ∠ BEC = double the whole ∠ BAC.

CASE II. Let the centre of the ⊙ be without the ∠ BAC.

Then it may be demonstrated, as in the first case,

that the ∠ FEC is double the ∠ FAC,

and that the ∠ FEB, a part of the first, is double the ∠ FAB, a part of the other,

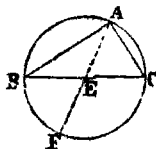
∴ the remaining ∠ BEC is double the remaining ∠ BAC.

Wherefore, *the angle at the centre*, &c. Q. E. D.

OBS In Fig 2 Case 1, the sum of the angles BEF, FEC or the angle BEC is a reflex or re-entrant angle

[See Notes, Book I, Def. 10.

NOTE If BE EC be in a straight line, the ∠ BEC is a straight angle (See Notes, Book I, Def 10), and is equal to two right angles, therefore BAC is a right angle, the figure BAC is a semicircle

[Compare Prop 31.

EXERCISE

From any point in a tangent to a circle a secant is drawn passing through the centre of the circle, and the angle between these two straight lines is bisected by a straight line which cuts the chord joining the extremity of the secant to the point of contact of the tangent, prove that the angle between the last two lines is half a right angle.

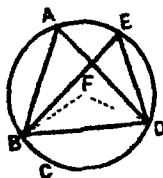
Proposition 21. Theorem.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED.

then the angles BAD, BED, *shall be equal to one another.*

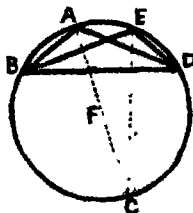
Take F the centre of the ⊙ ABCD. [III. 1.

First, let the segment BAED be greater than a semicircle. Join BF, DF.

Then, the ∠ BFD at the centre = double the ∠ BAD at the $\bigcirc^{ce}$, standing on the same arc BCD [III 20.

Similarly, the ∠ BFD = double the ∠ BED

∴ the ∠ BAD = the ∠ BED

Secondly, let the segment BAED be not greater than a semicircle

Draw AF to the centre, and produce it to meet the $\bigcirc^{ce}$ at C, and join CE

Then the segment BAEC is greater than a semicircle ;

∴ the ∠ s BAC, BEC, in it are equal, by the first case.

For the same reason, because the segment CAED is greater than a semicircle, the ∠s CAD, CED are equal.

∴ the whole ∠ BAD = the whole ∠ BED. [*Ax.* 2.

Wherefore, *the angles in the same segment*, &c. Q. E. D.

EXERCISES

1 If two equal angles stand on the same arc, the vertices of the angles will lie on the opposite segment

2 If two triangles on the same base, and on the same side of it have equal vertical angles, the extremities of the base and the vertices are concyclic

3 If two chords cut each other, and triangles be formed by joining their ends, the opposite pair of triangles are equiangular.

4 If innumerable triangles be constructed on the same base with equal vertical angles, the vertices would form the segment of a circle

5 A circle is circumscribed about an equilateral triangle ABC, and any point D is taken in the arc BC, show that the line DA is equal to the sum of DB and DC

6 Given the base and the opposite vertical angle of a triangle, find the locus of the intersection of the perpendiculars

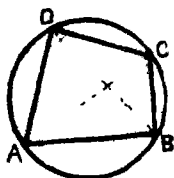
7. If AB be a fixed chord in a circle, and P any point on the circumference, then the internal bisector of the angle APB always passes through a fixed point

8 AB is the chord of a segment, and P any point on the arc; and the internal bisectors of the angles PAB, PBA, intersect at O. Find the locus of O

Proposition 22. Theorem.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD;

then any two of its opposite angles shall be together equal to two right angles.

Join AC, BD

The ∠ CAB = the ∠ CDB,

being in the same segment CDAB [III. 21.

Also, the ∠ ACB = the ∠ ADB,

being in the same segment ADCB [III. 21.

∴ the ∠ ADC = the sum of the ∠s CAB and ACB

To each equal add the ∠ ABC

∴ the sum of the ∠s ADC, ABC

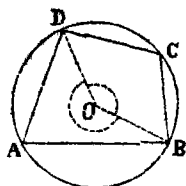
= the sum of the ∠s CAB, ACB, and ABC,

= two right angles. [I. 32.

In the same manner it may be shewn

that the sum of the ∠s BAD, BCD = two right angles.

Wherefore, *the opposite angles*, &c. Q. E. D.

Alternative Proof.

Take O the centre of the ⊙ ABCD
Join DO, BO

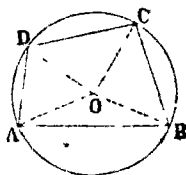
The ∠ BOD is double the ∠ BAD. [III, 20.

Also the re-entrant ∠ DOB is double the ∠ DCB [III 20.

∴ the sum of the central ∠s BOD, DOB = double the sum of the ∠s BAD, DCB

But the ∠s at O together = four right angles,

∴ the sum of the ∠s BAD, DCB = two right angles Q. E. D

ANOTHER PROOF

Take O the centre of the ⊙ ABCD.

Join OA, OB, OC, OD

The △s AOB, BOC, COD, DOA, are isosceles

∴ the ∠ OAB = the ∠ OBA,

and the ∠ OAD = the ∠ ODA,

∴ the whole ∠ BAD = the ∠s OBA, ODA

Similarly the ∠ BCD = the ∠s OBC, ODC

∴ the ∠s DAB, DCB together = the ∠s ABC, ADC

But the sum of all the angles of the quadrilateral is equal to four right angles, [I 32. *Cor.*

∴ the sum of the ∠s BAD, DCB = two right angles,

and the sum of the ∠s ABC, CDA = two right angles. Q E. D.

EXERCISES.

1. If in any quadrilateral the opposite angles be together equal to two right angles, the quadrilateral is cyclic.

2. If an irregular hexagon be inscribed in a circle, the first, third and fifth angles will together be equal to four right angles.

3. A rhombus cannot be inscribed in a circle.

4. A parallelogram inscribed in a circle is a rectangle.

5. Prove that the sum of the angles in the four segments of the circle, exterior to the inscribed quadrilateral, is equal to six right angles.

6. If each pair of opposite sides of a quadrilateral inscribed in a circle meet each other when produced, the straight lines bisecting the angles made between them are at right angles to each other.

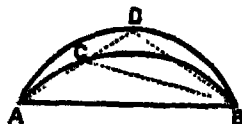
7. If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles, together with two right angles, is equal to as many right angles as the figure has sides.

8. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet in A, B, and about the triangles so formed without the quadrilateral, circles be described cutting again at C, show that A, C, B will be collinear.

9. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

10. If a figure of any even number of sides be inscribed in a circle, the sum of its alternate angles is equal to half the sum of all the angles of the figure.

Proposition 23. Theorem.

On the same chord, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB, not coinciding with one another.

Then, since the arcs ACB, ADB, intersect at A and B,

∴ they cannot intersect at any other point; [III. 10.

∴ one of the segments must fall within the other.

Let ACB fall within ADB

Take any point C on the inner arc;

join AC, and produce it to meet the outer arc at D.

Join BD, BC.

Then, because the segments are similar, [*Hyp.*

∴ the ∠ ACB = the ∠ ADB [III. *Def.* 12

that is, the ext angle of the △ BCD = the int angle, which is absurd [I 16.

Wherefore, *on the same chord*, &c Q. E. D.

Proposition 24 Theorem.

Similar segments of circles on equal chords are equal to one another

Let AEB, CFD be similar segments of circles on the equal chords AB, CD,

then the segment AEB *shall be equal to the segment* CFD.

For, if the segment AEB be applied to the segment CFD, so that A falls on C, and AB falls along CD,

then B must coincide with D,

for AB = CD.

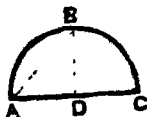
∴ the chord AB coinciding with the chord CD,

the segment AEB must coincide with the segment CFD, [III. 23.

and is therefore equal to it.

Wherefore, *similar segments*, &c. Q. E. D.

Proposition 25. Problem.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment of a circle; *it is required to describe the circle of which it is a segment.*

*Euclid's Solution.**

Bisect AC at D, [I. 10
from D draw DB at right angles to AC, [I. 11
and join AB.

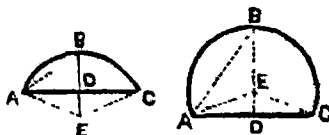
Then, the ∠s DAB, DBA, are either equal to one another, or they are not.

CASE 1. Let them be equal.

∴ BD = AD = DC.

∴ D is the centre of the ⊙ of which ABC is the arc. [III. 9

Hence the required ⊙ has D for centre and any of these lines for radius.†

CASE 2. Let the ∠s DAB, DBA, be not equal.

Then, at A make the ∠ BAE = the ∠ B, [I. 23
and let AE meet BD (produced if necessary) at E.
Join EC.

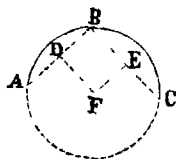
---

* Students reading this Book for the first time may omit the solution of Euclid and read the Alternative Solution.

† Note that the segment ABC is a semi circle, since AC is a diameter.

Now, since the $\angle$ BAE = the $\angle$ ABE,
∴ EA = EB. [I. 6.
Also, in the $\triangle$s EDA, EDC,
DA = DC, [Cons.
ED is common,
and the $\angle$s at D are equal, being right angles ·
∴ EA = EC [I. 4.
∴ EA = EB = EC
∴ E is the centre of the ⊙ of which ABC is the arc [III. 9
Hence the required ⊙ has E for centre and any of these lines for radius *

Wherefore, *a segment of a circle being given, the circle has been described of which it is a segment* Q. E. F.

Alternative Solution

Take any point B in the arc ABC,
and join AB, BC

Draw DF bisecting AB at right angles, [I 10 11.
and draw EF bisecting BC at right angles

Then because FD bisects the chord AB at right angles,
∴ the centre of the ⊙ lies in FD [III. 1. Cor.

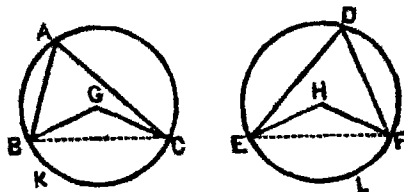
Likewise, the centre of the ⊙ lies in FE
∴ F, their point of intersection, is the centre of the ⊙ required.

∴ the circle described from F as centre and with FA, FB, or FC, as radius, will be the required circle Q. E. F.

---

* It is evident that if the $\angle$ ABD be greater than the $\angle$ BAD,
the centre E falls without the segment ABC,
∴ the segment is less than a semicircle
But if the $\angle$ ABD be less than the $\angle$ BAD,
the centre E falls within the segment ABC,
∴ the segment is greater than a semicircle.

Proposition 26. Theorem.

In equal circles equal angles stand on equal arcs, whether the angles be at the centres or the circumferences.

Let ABC, DEF be equal circles; and let BGC, EHF be equal angles in them at their centres, hence let BAC, EDF be equal angles at their circumferences [III 20.

then the arc BKC *shall be equal to the arc* ELF

Join BC, EF

Then, the ⊙s being equal,

their radii are equal

Hence, in the △s BGC, EHF,

GB = HE,

GC = HF,

and the ∠ G = the ∠ H

∴ BC = EF [I. 4.

And because the ∠ A = the ∠ D, [*Hyp.*

∴ the segment BAC is similar to the segment EDF, [III *Def.* 12

and they are on equal chords BC, EF

∴ the segment BAC = the segment EDF [III 24.

But the whole ⊙ ABC = the whole ⊙ DEF, [*Hyp.*

∴ the remaining segment BKC = the remaining segment ELF,

∴ the arc BKC = the arc ELF

Wherefore, *in equal circles,* &c Q E D.

COR In the same circle, equal angles stand upon equal arcs whether they be at the centre or at the circumference

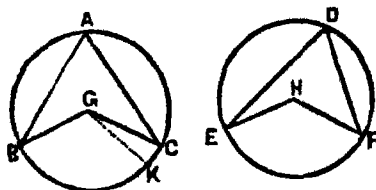
EXERCISES

1 Parallel chords in a circle intercept equal arcs

2. If the opposite angles of a quadrilateral in a circle be equal, the diagonal opposite to them must be a diameter.

3 If two chords of a circle intersect at right angles, the sum of the arcs they intercept is a semicircle.

Proposition 27. Theorem.

In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or the circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres, and the angles BAC, EDF at their circumferences, stand on equal arcs BC, EF,

then the angle BGC *shall be equal to the angle* EHF, *and consequently the angle* BAC *equal to the angle* EDF.

Then, if the ∠s BGC, EHF, are not equal, one must be greater than the other.

If possible, let BGC be the greater.

At G, in BG, make the ∠ BGK = the ∠ EHF. [I. 23.

Then, because the ⊙ BAC = the ⊙ EDF,
and the ∠ BGK = the ∠ EHF,
∴ the arc BK = the arc EF. [III. 26.
But the arc EF = the arc BC, [Hyp.
∴ the arc BK = the arc BC:
the part equal to the whole, which is absurd.
∴ the ∠ BGC is not unequal to the ∠ EHF,
that is, it is equal to it.

And the ∠ A = half the ∠ BGC, [III. 20.
also, the ∠ D = half the ∠ EHF.
∴ the ∠ A = the ∠ D.

Wherefore, *in equal circles,* &c. Q. E. D.

COR. In the same circle, the angles which stand upon equal arcs are equal, whether they be at the centre or at the circumference.

Alternative Proof.

Apply the ⊙ BAC to the ⊙ EDF,
so that the centre G may fall on the centre H,
and the radius BG on the radius EH.
Then, the ⊙s being equal,
the ○ces would coincide, and the radii BG, EH, would coincide.
Also, the arcs BC, EF, being equal,
would coincide, so that C would fall on F
∴ GC would coincide with HF
∴ the ∠s BGC, EHF coinciding, would be equal to one another
∴ the ∠A = the ∠D,
being halves of the ∠s BGC and EHF, respectively [III. 20

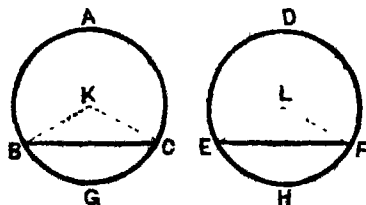
NOTE This, being a direct proof, is more rigorous than that of Euclid

EXERCISES

1 The chords which intercept equal arcs of a circle, are parallel

2 In equal circles, sectors which stand upon equal arcs are equal

Proposition 28. Theorem.

In equal circles, equal chords cut off equal arcs, the greater arc equal to the greater, and the less equal to the less.

Let ABC, DEF, be equal circles, and BC, EF, equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF

then the greater arc BAC *shall be equal to the greater arc* EDF, *and the less arc* BGC *equal to the less arc* EHF.

Take K, L, the centres of the ⊙s. [III. 1
and join BK, KC, EL, LF.

Then, because the ⊙s are equal,
∴ their radii are equal

Hence in the $\triangle$s BKC, ELF,
BK = EL,
KC = LF,
and BC = EF : [Hyp.
∴ the ∠ BKC = the ∠ ELF, [I. 8.
∴ the arc BGC = the arc EHF. [III. 26.
But the whole ○ce BACG = the whole ○ce EDFH,
∴ the remaining arc BAC = the remaining arc EDF.

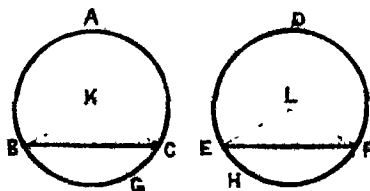
Wherefore, *in equal circles*, &c Q E D.

COR. In the same circle, equal chords cut off equal arcs, the greater arc equal to the greater, and the less equal to the less

EXERCISE.

Through a given point to draw a straight line which shall cut off a given arc from a given circle

Proposition 29 Theorem

In equal circles equal arcs are subtended by equal chords

Let ABC, DEF be equal circles,
and let BGC, EHF be equal arcs in them,
then the chord BC *shall be equal to the chord* EF.
Take K, L, the centres of the circles, [III. 1.
and join BK, KC, EL, LF.

Then, since the ⊙ ABC = the ⊙ DEF,
and the arc BGC = the arc EHF :
∴ the ∠ BKC = the ∠ ELF [III 27.
Hence, in the $\triangle$s BKC and ELF,
the radius BK = the radius EL,
and the radius KC = the radius LF,
and the ∠ BKC = the ∠ ELF.
∴ BC = EF. [I. 4.

Wherefore, *in equal circles* &c. Q. E. D.

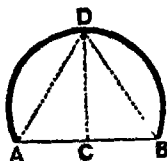
COR In the same circle, equal arcs are subtended by equal chords

EXERCISES.

1. The straight lines which join the extremities of two parallel chords in a circle are equal

2 In a circle, the straight lines which intercept equal arcs are parallel.

Proposition 30 Problem

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc : it is required to bisect it

Join AB ,

bisect it at C , [I 10

from C draw CD at right angles to AB, meeting the arc at D. [I 11.

Then the arc ADB *shall be bisected at* D.

Join AD, DB

Then, in the △s ACD, BCD,

AC = CB,

CD is common,

and the ∠s at C are equal, being right angles

∴ AD = DB [I. 4.

But equal chords cut off equal arcs, the greater arc equal to the greater, and the less arc equal to the less , [III 28 *Cor.* and each of the arcs AD, DB, is less than a semi-circumference, because DC, if produced, is a diameter , [III 1, *Cor.*

∴ the arc AD = the arc BD

Wherefore, *the given arc is bisected at* D. Q. E. F.

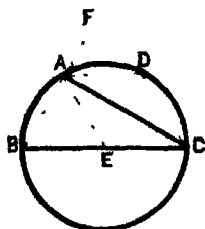
EXERCISES.

1 The bisectors of the vertical angles of all triangles having a given base and inscribed in a circle, are concurrent at a point on the circumference.

2. The bisectors of the external vertical angles of all triangles having a given base and inscribed in a circle, are concurrent at a point on the circumference.

Proposition 31 Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle

Let ABCD be a circle, of which BC is a diameter, and E the centre; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC,

(1) *then the angle in the semicircle* BAC *shall be a right angle*

(2) *but the angle in the segment* ABC, *which is greater than a semicircle, shall be less than a right angle*,

(3) *and the angle in the segment* ADC, *which is less than a semicircle, shall be greater than a right angle*

Join AE, and produce BA to F

(1) Then, because EA = EB,

∵ the ∠ EAB = the ∠ EBA [I 5.

Also, because EA = EC,

∴ the ∠ EAC = the ∠ ECA:

∴ the whole ∠ BAC = the sum of the ∠s EBA and ECA.

But the ext ∠ FAC = the sum of the int. ∠s CBA and BCA · [I. 32.

∴ the ∠ BAC = the ∠ FAC;

∴ each of them is a right angle; [I. *Def.* 11.

∴ the ∠ BAC in the semicircle BAC is a right angle.

(2) In the △ ABC,

∵ the ∠ BAC = a right angle,

∴ the ∠ ABC is less than a right angle, [I. 17.

∴ the ∠ in the segment ABC, which is greater than a semicircle, is less than a right angle

(3) Finally, because ABCD is a quadrilateral inscribed in the ⊙,

∴ the sum of the ∠s ABC and ADC = two rt. angles; [III. 22

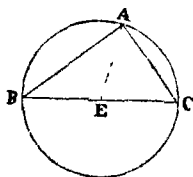
But the ∠ ABC is less than a right angle,

∴ the ∠ ADC is greater than a right angle,

∴ the ∠ in the segment ADC, which is less than a semicircle, is greater than a right angle

Wherefore, *in a circle, the angle &c* Q. E. D

COR From the demonstration it is manifest that if one angle of a triangle be equal to the other two, it is a right angle For the angle adjacent to it is equal to the same two angles, [I 32. and when the adjacent angles are equal, they are right angles

Alternative Proof

Let BAC be the segment.

(1) Let BAC be a semi-circle,
let BC pass through the centre E
Join AE

The ∠ AEC is double the ∠ B [III 20.

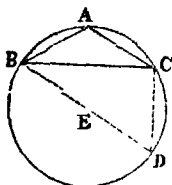
But the ∠ ABE = the ∠ BAE, (∵ AE = BE)

∴ the ∠ AEC is double the ∠ BAE.

Similarly, the ∠ AEB is double the ∠ EAC

∴ the sum of the ∠s AEC, AEB or two right angles
= double the sum of the ∠s BAE, EAC, or the ∠ BAC.

∴ BAC is a right angle.

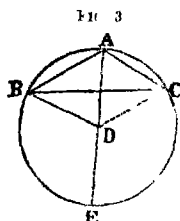
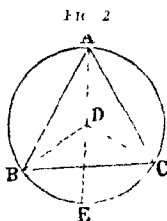
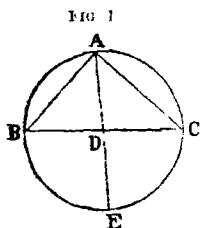
(2) and (3) Let the segment BDC be greater than a semi-circle, and the segment BAC less than a semi-circle, and let BD be a diameter

The ∠ BCD is a right angle by (1)

∴ the ∠ D is less than a right angle [I 17.

But the sum of the ∠s D and A = two right angles [III 22.

∴ the ∠ BAC is greater than a right angle

Wherefore, *in a circle*, &c Q E D

ANOTHER PROOF

Let BAC be the segment Find D the centre Join AD and produce it to meet the ⊙ce again at E Join BD, DC

In Fig 1 the segment is a semicircle, and BC is a diameter,

∴ the centre is in BC

In Fig 2, D is above BC

In Fig 3, D is below BC

In all these figures, the central ∠ BDC is double the ∠ BAC.

In Fig 1, the ∠ BDC is a straight angle,

and ∴ = two right angles (See *Notes* Book I *Def* 10),

∴ BAC is a right angle

In Fig 2 the ∠ BDC is less than two right angles,

∴ the ∠ BAC is less than a right angle

In Fig 3, the ∠ BDC is a re-entrant angle,

∴ greater than two right angles;

∴ BAC is greater than a right angle

EXERCISES.

1 The chord which passes through the centre is a diameter; hence any diameter of a circle is its axis of symmetry

2. From a point without a circle two equal secants are drawn; shew that the diameter through the point is the axis of symmetry of the two segments cut off by the secants

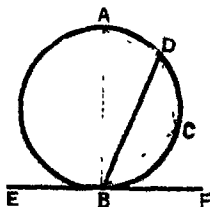
3. If a circle be described on the radius of another circle as diameter, any chord in the latter, drawn from the point in which the circles meet, is bisected by the former

4. Draw two tangents to a circle from a point outside the circle

5. If two chords of a circle cut each other at right angles, the sum of the squares of the segments is equal to the square on the diameter

Proposition 32 Theorem.

If from the point of contact of a tangent to a circle a straight line be drawn cutting the circle, the angles which this line makes with the tangent shall be equal to the angles which are in the alternate segments of the circle

Let EF be a tangent to the circle ABCD at the point B, and from B let the straight line BD be drawn, cutting the circle:

then the angles which BD *makes with the tangent* EF, *shall be equal to the angles in the alternate segments of the circle*,

that is, the angle DBF *shall be equal to the angle in the segment* BAD,

and the angle DBE *shall be equal to the angle in the segment* BCD.

From the point B draw BA at right angles to EF, meeting the ☉ce at A, [I. 11.

take any point C in the arc BD, and join AD, DC, CB.

Then, since AB is ⊥ to the tangent EF at the point of contact B,

∴ the centre of the ⊙ lies in AB, [III. 19.
∴ the ∠ADB = a right angle, [III. 31.
∴ in the △ ADB,
the sum of the other ∠s DAB, DBA = a rt. angle, [I. 32.
= the ∠ABF.

Take away the common part, the ∠DBA,
∴ the ∠DBF = the ∠DAB,
which is in the alternate segment.

But the ∠DBE = the supplement of the ∠DBF,

and the ∠DCB = the supplement of the ∠DAB, [III. 22.
since ABCD is a quadrilateral inscribed in the ⊙.

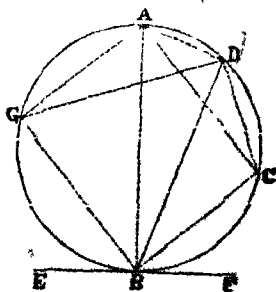
∴ the ∠DBE = the ∠DCB,
which is in the alternate segment.

Wherefore, *if from the point of contact, &c* Q. E. D.

Alternative Proof

Make the same construction as before, take any point G in the semi-circle AGB,
join AG, GB, GD, AC.

The ∠AGB is a right angle, [III. 31.
∴ the ∠AGB = the ∠ABF
But the ∠AGD = the ∠ABD [III. 21.
∴ the remaining ∠DGB = the remaining ∠DBF [Ax. 3.
Likewise, the ∠ACB is a right angle, [III. 31.
and it is equal to the ∠ABE,
also the ∠ACD = the ∠ABD [III. 21.
∴ the whole ∠DCB = the whole ∠DBE. [Ax. 2.

Q. E. D.

ANOTHER PROOF.

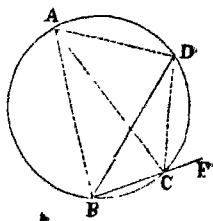
Let BCF be a secant

The ∠ DBC or DBF = the ∠ DAC. [III 21

In the arc BC, let the point C move towards B until it becomes consecutive to B

Then BF will be a tangent at B (See *Notes* Book III Def 3), and AC will coincide with AB, and DC with DB, then the ∠ DAC will coincide with the ∠ DAB

∴ the ∠ DBF will be equal to the ∠ DAB, which is in the alternate segment

Q E D

Otherwise

The sum of the ∠s DAB, DCB = two rt angles [III 22.

But the sum of the ∠s DCB, DCF = two rt angles [I 13.

Take away the common ∠ DCB

∴ the remaining ∠ DAB = the remaining ∠ DCF

Now let the point C move towards B till C becomes consecutive to B Then BF will be a tangent and DC will coincide with DB ∴ the ∠ DBF will be equal to the ∠ DAB

EXERCISES

1 If several circles touch one another at the same point either internally or externally any straight line passing through the point of contact will cut off similar segments from them

2 If one circle touch another either internally or externally, and if two straight lines be drawn from or through the point of contact cutting them both, the chords joining the points of intersections are parallel

3 If a tangent of a circle be parallel to a chord, the arc intercepted by the chord is bisected at the point of contact

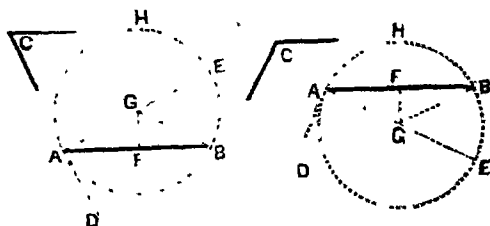
4 Tangents through the extremities of the same chord make equal angles with it on the same side

5 The chords joining the points of contact of two parallel tangents is a diameter

Proposition 33 Problem.

On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle

Let AB be the given straight line, and C the given rectilineal angle ,

It is required to describe on the given straight line A B, a segment of a circle containing an angle equal to the angle C.

At A in BA make the ∠ BAD = the ∠ C [I 23.

Draw AE at right angles to AD [I 11.

Let FG bisect AB at right angles [I. 10. 11.

meeting AE at G

Join GB

Then in the △s AFG, BFG,

AF = BF

FG is common,

and the ∠s at F are equal, being right angles .

∴ AG = BG

∴ the ⊙ described with centre G, and radius GA, will pass through B

Let this ⊙ be ABH

Then the ∠ in the segment AHB = *the given ∠* C.

Since AE is a diameter,

and AD is at rt angles to AE at its extremity A,

∴ AD is a tangent to the ⊙. [III 16. *Cor.*

And, since AB is a chord through its point of contact,

∴ the ∠ BAD = the ∠ in the alternate segment AHB. [III. 32.

But the ∠ BAD = the ∠ C ;

∴ the ∠ in the segment AHB = the ∠ C.

Wherefore, *on the given straight line* AB, *the segment* AHB *of a circle has been described, containing an angle equal to the given angle* C. Q. E. F.

OBS. If the given angle C is a right angle, then the solution is simpler. For the semicircle described on AB as diameter is the required segment [III 31.

Alternative Solution

On AB describe an isosceles △ AGB,

(1) whose base angle = the complement of the ∠ C,
if the ∠ C is less than a right angle

(2) whose base-angle
= the complement of the supplement of the ∠ C,
if the ∠ C is greater than a right angle

With G as centre and GA or GB as radius, describe a ⊙.

Then, each base-angle of the △ AGB
= the complement of the ∠ in the major segment on AB [III 31

∴ (1) the complement of the ∠ in the major segment on AB
= the complement of the ∠ C,

∴ the ∠ in the major segment on AB = the ∠ C

∴ (2) the complement of the ∠ in the major segment on AB
= the complement of the supplement of the ∠ C,

∴ the ∠ in the major segment on AB = the supplement of the ∠ C.

∴ the ∠ in the minor segment on AB = the ∠ C [III 22.

Q. E. F.

EXERCISES.

1. Given the base and the vertical angle of a triangle, construct the locus of the vertex

2 Given the base, the vertical angle and the perpendicular from the vertex on the base, construct the triangle

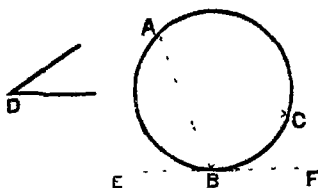
3. Given the base, the vertical angle, and the sum or difference of the sides, construct the triangle

4. Given the base, the vertical angle, and the perpendicular from one extremity of the base on the opposite side, to construct the triangle

Proposition 34. Problem.

From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off from the circle ABC a segment containing an angle equal to the angle D.

At any point B on the $\bigcirc^{ce}$ draw the tangent EBF [III. 17.

Through B draw the chord BC,
making the $\angle$ FBC = the $\angle$ D [I. 23.

Then the segment BAC *shall contain an angle* = *the* $\angle$ D.

For, since BF is a tangent, and BC a chord through the point of contact,

$\therefore$ the $\angle$ FBC = the $\angle$ in the alternate segment BAC. [III. 32.

But the $\angle$ FBC = the $\angle$ D,
$\therefore$ the $\angle$ in the segment BAC = the $\angle$ D

Wherefore, *from the given circle* ABC, *the segment* BAC *has been cut off, containing an angle equal to the given angle* D.

Q. E. F.

EXERCISES

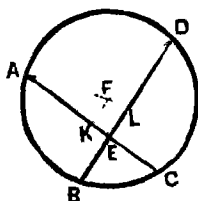
1 Through a given point to draw a straight line that shall cut off, from a given circle, a segment containing a given angle.

2. Given an angle at the base, the vertical angle, and the radius of the circumscribing circle, construct the triangle.

Proposition 35 Theorem.

If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, cut one another at the point E, within the circle ABCD,

then the rectangle contained by AE, EC *shall be equal to the rectangle contained by* BE, ED

Find F the centre of the ⊙ [III. 1.
Join FA, FD FE,
and draw FK FL perp to AC BD, respectively
Then, since FK is drawn from the centre perp to AC,
∴ AC is bisected at K [III. 3.
Similarly BD is bisected at L
Now, since AC is divided equally at K and unequally at E,
∴ the rect AE, EC with the sq on KE = the sq on AK.
[II. 5.
To each add the sq on FK
∴ the rect AE, EC with the sqs on KE, FK
= the sqs on AK, FK
∴ the rect. AE, EC with the sq on FE = the sq on AF.
[I. 47.
Similarly we may shew that
the rect BE, ED with the sq. on FE = the sq on DF.
But the sq on AF = the sq on DF, [AF, DF being radii];
∴ the rect AE, EC with the sq on FE
= the rect. BE, ED with the sq on FE.
Take away the common part the sq on FE ·
∴ the rect AE, EC = the rect. BE, ED.

Wherefore, *if two straight lines*, &c. Q. E. D.

Obs In the following particular cases the proof of this Prop. becomes much simpler, which the student might work out for himself —

(1) When both the given straight lines pass through the centre.

(2) When one passes through the centre and is perp to the other

(3) When one passes through the centre and meets the other obliquely

Cor It follows from this Prop that if any number of chords intersect at a given point within a circle, the rectangles contained by the segments of each chord are all equal

Exercises

1 If two straight lines AB CD intersect in E, and if the rectangle contained by AE EB be equal to the rectangle contained by CE ED, the four points A, B, C D are concyclic

2 In a circle, the rectangle contained by the segments of a chord, is equal to the difference of the squares of the radius and of the straight line joining the centre with the point of section

3 In a circle, if a perpendicular to a diameter be drawn from any point in a chord the rectangle contained by the segments of the diameter is equal to the rectangle contained by the segments of the chord, together with the square on the perpendicular

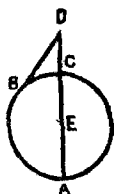
4 If two triangles be equiangular the rectangle contained by any side in one triangle, and the side about the equal angle in the other, which is opposite to the angle not equal to that which is opposite to the former side, is equal to the rectangle contained by the remaining sides about the equal angles

5 If from the vertical angle of a triangle, a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the perpendicular and the diameter of the circumscribing circle

Proposition 36 Theorem.

If from any point without a circle a secant and a tangent be drawn then the rectangle contained by the whole secant and the part of it without the circle is equal to the square on the tangent

Let D be any point without the circle ABC, and let DCA be a secant and DB a tangent, drawn through D

then the rectangle AD, DC *shall be equal to the square on* DB.

First Let DCA pass through the centre E

Join EB

Then, since AC is bisected at E and produced to D,

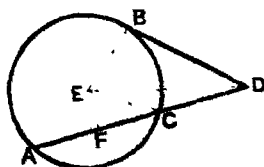
∴ the rect AD, DC with the sq on EC
= the sq on ED [II 6.
= the sqs on EB, DB [I 47.

But the sq on EC = the sq on EB

∴ the rect, AD, DC with the sq on EB
= the sqs. on EB, DB.

Take away the common part, the sq on EB

∴ the rect AD, DC = the sq on DB

Secondly Suppose DCA not to pass through the centre E.

Draw EF perp to AD, and join EB, EC, ED.

Then, since EF passing through the centre is perp. to the chord AC, which does not pass through the centre,

∴ AC is bisected at F [III. 3.

Then, since AC is bisected at F and produced to D,

∴ the rect AD, DC with the sq on FC = the sq on FD. [II. 6.

To each add the sq on EF

∴ the rect AD, DC with the sqs. on EF, FC
= the sqs on EF, FD

∴ the rect AD, DC with the sq on EC
= the sq on ED, [I 47.
= the sqs on EB, DB [I 47.

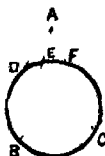
But the sq on EC = the sq on EB

∴ the rect AD, DC with the sq on EB
= the sqs on EB, DB

Take away the common part, the sq on EB

∴ the rect AD, DC = the sq on DB

Wherefore, *if from any point*, &c Q E D.

COR If from any point without a circle, there be drawn any number of secants, as for instance AB, AC, the rectangles contained

by the whole secants and the parts of them without the circle are all equal, namely the rectangle BA, AE is equal to the rectangle CA, AF, etc , for each of them is equal to the square on the tangent AD

NOTE If we bear in mind that two chords of a circle may intersect either *internally* at a point within the circle, or *externally* at a point without the circle when produced to meet, and if we follow the definition of the internal or external segments of a straight line (Introduction to Book II p 144), then the Corollaries of Props. 35 and 36 may be included in one general Prop —

If any number of chords of a circle intersect at a fixed point either internally or externally, then the rectangles contained by the segments of each of these chords are all equal.

OBS It might be observed that Prop 36 is only a particular case of this general enunciation when the point of section is external For in that case the tangent is only the limiting position of the secant through the external point, that is, when the external segments of the chord are equal

EXERCISES

1 If two fixed straight lines produced meet at a point, and if the rectangle contained by the whole of one line produced and the part of it produced, be equal to the rectangle contained by the whole of the other line produced and the part produced, the extremities of the given straight lines are concyclic

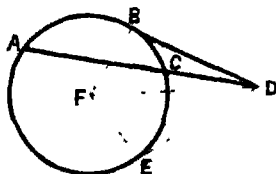
2 Two circles cut each other, find the locus of the point from which tangents to the circles are equal

3 If a given circle be cut by any number of circles, which all pass through two given points without it the straight lines which join the points of intersection, are either parallel or all meet, if produced, at the same point

Proposition 37 Theorem

If from any point without a circle there be drawn two straight lines one of which cuts the circle and the other meets it, and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall touch it

Let any point D be taken without the circle ABC, and from it let two straight lines DCA DB be drawn, of which DCA cuts the circle and DB meets it and let the rectangle AD, DC, be equal to the square on DB

then DB *shall touch the circle*

From D draw the tangent DE [III. 17.

Find F the centre, [III. 1.

and join FB, FD, FE.

Then, since DE is a tangent and DCA a secant,

∴ the rect. AD, DC = the sq. on DE. [III. 36.

But the rect AD, DC = the sq on DB [*Hyp.*

∴ the sq on DE = the sq on DB,

∴ DE = DB

Now in the △s DBF, DEF,

DB = DE,

BF = EF, being radii,

and DF is common:

∴ the ∠ DBF = the ∠ DEF [I. 8.

= a right angle,

since the ∠ DEF is a right angle DE being a tangent; [III 18.

∴ DB is also a tangent, since it is ⊥ the radius FB [III. 16.

Wherefore, *if from any point*, &c Q E D

EXERCISES

1. If two circles cut each other, and from any point in the common chord produced, tangents be drawn one to each circle, prove that these tangents are equal

2 If three circles touch one another externally, the tangents at the points of contact meet in one point

3 Describe a circle which shall pass through two given points and touch a given straight line

NOTES ON BOOK III.

The third Book of the Elements treats of the properties of circles

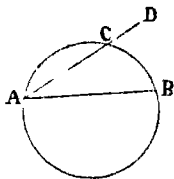
Def 1 This is a theorem, not a definition, and it can easily be demonstrated, for when the centres coincide, if the radii be equal, the circumferences must coincide, whence it follows that the circles are equal

Def 2 Every chord, except a diameter, divides a circle into two unequal segments, the one greater and the other less than a semi-circle

Def 3 A straight line which touches a circle, is called a *tangent* to the circle and a straight line which cuts a circle, is called a *secant* The point in the circumference of a circle where a straight line or another circle *touches* it, is called *the point of contact*

In Modern Geometry, a curve is supposed to be made up of an infinite number of points, placed one after the other along the curve,

and the straight line passing through any two consecutive points on a curve is a ***tangent*** *to the same*

Let ACB be a circle Let the straight line AD, passing through A, C, be a secant

Let the point C move along the circumference of the circle towards A When the point C is next to A the line AD will be a tangent Hence *a tangent is the limiting position of a secant* In this position, the two points A, C, having no dimensions, will virtually come to one point, hence the definition of Euclid But the definition of Modern Geometry is preferable, as it leads to several important results (*See* Alternative Proofs Props 16, 32)

Def 4 One circle may touch another *externally* or *internally.*

Two circles touch each other *externally* when the convex circumference of one circle touches the convex circumference of the other, but one circle touches another circle *internally* when the

convex circumference of the one touches the concave circumference of the other Hence for internal contact one circle must be smaller than the other

Def. 8 This definition is not used in the Elements

Def 9 This definition may be expressed thus —if from the extremities of the chord of a segment, two other chords be drawn to any point in the arc of the segment, the angle formed between these two chords is called *the angle in the segment.*

Def 11 The two radii divide a circle into two unequal sectors, which become equal when the two radii are in the same straight line and thus make a diameter A sector becomes a *quadrant*, or the fourth part of a circle, when the radii are at right angles to each other a *sextant* or sixth part when the angle between the radii is two thirds of a right angle, an *octant*, or eighth part, when the angle between the radii is half a right angle

Prop 1 A straight line is said to be drawn in a circle when its extremities are terminated by the circumference of the circle In the construction CD is produced to E, this assumes that D is within the circle, which Euclid proves in III 2

The simplest practical mode of finding the centre of a circle, is to draw any two chords, and to bisect them at right angles by two straight lines The intersection of these straight lines will be the centre (III 1 *Cor*)

Prop 2 Candala has given a direct proof of this Proposition, which Euclid seems to have rejected, because the principle on which it depends, namely, that the *extremity of a straight line less than the radius is within the circle* does not appear among the axioms, and he preferred using an indirect proof to increasing the number of axioms

Prop 3 is the converse of the Corollary to III. 1

Prop 4 The only case in which the two chords can bisect each other, is when they pass through the centre, that is, when they are both diameters

Props 5 6 The demonstrations of these two Propositions are precisely the same, and they should have been combined in one Proposition —*If the circumferences of two circles meet, they cannot have the same centre*

Props 7 8. These two Propositions are essentially the same, the only difference being that in the former the point is taken within the circle, and in the latter, without the circle

Hence, these Propositions may be enunciated and proved jointly, in the same manner as in the Alternative Proof of Prop. 8.—

Of all straight lines which can be drawn from any point, within or without a circle, to the circumference, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre, and of the rest, the one which subtends a greater angle at the centre is greater than one which subtends a less angle, and from the same point not more than two equal straight lines can be drawn to the circumference, one on each side of the line joining the centre with the given point

Each of these straight lines may be considered to be the base of a series of △s, whose sides are composed of the fixed straight line joining the point with the centre, and the radius to the other extremity of the base Now in all these △s, the two sides would be identically equal, each to each since the line joining the point with the centre would be common to them all, and the radii to the other extremity of the bases would be equal Hence the magnitude of the bases would depend upon the angles they subtend at the vertices of the △s, that is, at the centre of the ⊙ (I 24)

∴ those straight lines from the fixed point to the ○ce, which subtend a greater angle at the centre, are greater than those which subtend a less angle,

also, that which passes through the centre subtending the greatest possible angle (two right angles), is the greatest,

and that which when produced passes through the centre, subtending the least angle (the angle vanishing altogether), is the least, etc

Prop 8 An arc of a circle is said to be concave towards a point without it, when all the straight lines drawn from that point, meet the *hollow part* or *inside* of the arc, and it is said to be *convex* towards a point, without it when all the straight lines, drawn from that point meet the *round part* or *outside* of the arc

Prop 9 In the demonstration of this proposition all the three straight lines DA DB and DC are supposed to lie on one side of the diameter FG But the point E might be so chosen that DC and DB should be on opposite sides of FG, and then DC might be equal to DB instead of being greater than it In such a case however the absurdity would follow that at least two of the given straight lines are unequal

Props 11, 12 In the enunciations of these Propositions, only one point is assumed to be the point of contact that two circles have only one point of contact is proved in III 13

Prop 13 The following is Euclid's demonstration of the first case of this Proposition - If possible let the circle EBF touch the circle ABC on the inside at the points B and D Let P be the centre of the circle ABC, and Q the centre of the circle EBF Join PQ, then PQ produced will pass through B and D Thus BPQD will be the diameter of both the circles, therefore BPQD will be bisected at the two points P and Q, which is absurd.

Prop. 14 The *distance* of a chord from the centre is the perpendicular drawn from the centre to the chord.

Prop. 17. When the given point is without the circumference of the given circle, it is obvious that two equal tangents can be drawn from the given point to touch the circle —

In the point E and in the straight line AE make the angle AEH equal to the angle AEB, EH cutting the circumference of the circle BCD in H Join AH, then AH is also a tangent drawn from A, and equal to AB

By I 4, we can prove that the angle EHA is equal to the angle EBA and therefore it is a right angle, wherefore AH is the other equal tangent to the circle

By the help of III 31, the problem may be solved in another way .— Describe a circle on AE as diameter cutting the circle CDB at B and H Then straight lines drawn from A to B and H will be the two equal tangents from A

[Lacroix's *Éléments de Géométrie*]

Prop 18 is the converse of Proposition 16

Prop 20 Euclid has here omitted the case, when any of the straight lines which contain the angle at the circumference passes through the centre of the circle In this case the proof follows immediately from I 5 and 32

If the angle at the circumference reach or exceed a right angle, it is plain that the angle at the centre must reach or exceed two right angles, in the first case the two straight lines which make the angle at the centre become one straight line and in the second case the angle becomes a *re-entrant* angle, and it stands on the arc which the assumed arc wants of the whole circumference

In the demonstration of this Proposition two principles are assumed --(1) If two magnitudes be each double of two others, the sum of the former is double the sum of the latter (2) If two quantities be each double of two others, the difference of the two former is double of the difference of the two latter These assumptions are demonstrated in Book V 1 and 2

Prop 21 The following is the converse of this Proposition — The locus of the vertices of all triangles upon the same side of the same base, and which have the same vertical angle, is the circumference of the segment of a circle

Prop. 22. The converse of this Proposition is true, namely, *if the opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral* See Additional Prop III at the end of Book IV.

Prop 25 The proof of Euclid is cumbrous and at the same time defective It is assumed by Euclid that AE shall meet BD. That AE meets BD may be proved in the following manner :—

Because ADB is a right angle, therefore ABD is an acute angle (I 32) The angle BAE, which is made equal to ABD, is also an acute angle Therefore the sum of the angles ABD, BAE, is less than two right angles. Therefore AE shall meet BD (Ax. 12.)

***Props* 26 to 29. The properties proved in these four Propositions with respect to *equal circles* are also true with respect to the *same circle*.**

The student should remember the following —

In Prop. 26, the *angles* are given equal, the *arcs* are proved equal, and in Prop. 27, the *arcs* are given equal, the *angles* are proved equal ∴ Prop 27 is the converse of Prop 26.

In Prop 28, the *chords* are given equal, the *arcs* are proved equal, and in Prop 29, the *arcs* are given equal, the *chords* are proved equal ∴ Prop 29 is the converse of Prop 28

Prop 31 By this Proposition we may draw a straight line at right angles to a given straight line from one of its extremities without producing the line For if AB be the given straight line and A the given point, describe a circle passing through the points A, B Draw the diameter BC, join CA CA is at right angles to AB

Prop 32 The case in which the straight line cutting the circle passes through the centre may be easily proved, because the line then becomes a diameter and the angle which it makes with the tangent is a right angle [III 18], also the angle in the alternate segment which is a semicircle is also a right angle (III 31)

Prop 33 The centre G may also be found by making the angle GBA equal to the angle GAB

Prop 35 The following is the converse of the Corollary to this Proposition - *If any number of straight lines cut one another, and the rectangles contained by the segments of each be all equal, a circle may be described which shall pass through the extremities of these lines.*

[For a particular case, see Addl Prop IV at the end of Book IV]

Prop 36 The following is the converse of the Corollary to this Proposition *If any number of straight lines when produced meet at a point, and if the rectangles contained by each of these lines produced and its produced part be all equal, then a circle may be described passing through the extremities of the original lines*

[For a particular case, see Addl Prop. V at the end of Book IV.]

QUESTIONS ON BOOK III.

1. With what figures is Book III. of Euclid mainly taken up?

2. Define accurately each of the following:—*radius*, *arc*, *circumference*, *chord*, *tangent*, and *secant*.

3. Define *a sector of a circle*, *a segment of a circle*. When is a *sector* also a *segment*?

4. What are the points of resemblance and difference in the following:—chord and diameter, segment of a circle and a semicircle, a sector and a semicircle?

5. When does one circle touch another circle *internally*? Can two circles touch each other *internally*?

6. Define *quadrant*, *sextant*, and *octant*. When does *a sector* become a *quadrant*, *a sextant*, and *an octant* respectively?

7. What is meant by *cutting* and *touching* a circle as applied to lines and circles?

8. Distinguish accurately the angle of a segment and the angle in a segment.

9. How is the distance of a line in a circle from the centre measured?

10. What are similar segments of circles?

11. When is a straight line said to be placed in a circle?

12. Can a circle be drawn passing through three points in the same straight line?

13. If a straight line passing through the centre of a circle bisect a straight line in it, it shall cut it at right angles. Point out the exception.

14. When can two chords in a circle bisect each other?

15. How may the Propositions 5 and 6 be combined in one Proposition?

16. How many equal chords passing through any point in one circle can be drawn, the point not being the centre of a circle?

17. What is the shortest distance of a circle from a point out of it?

18. Is there any similarity between Propositions 7 and 8? State the difference.

19. When is an arc of a circle *concave* and when *convex* with reference to a point without it?

20. Two parallel chords in a circle are 8 and 10 inches in length and one inch apart? Find the distance of the larger one from the centre.

21. If one circle be contained within another without meeting it, show that the distance between their centres is less than the difference of their radii

22. Show that the shortest chord that can be drawn through a given point inside a circle is the chord which is perpendicular to the diameter passing through the given point

23 Show that the diameter is the longest straight line in a circle, and that other lines diminish as they recede from it

24 What is the locus of the middle points of all equal straight lines in a circle

25 Show that the tangents at the extremities of the diameter of a circle are parallel to each other.

26 The chords of an arc and of double the arc are 10 and 16 inches respectively, find the length of the diameter of the circle.

27. Two chords in a circle are 6 and 8 inches in length, the distance of the larger one from the centre is 3 inches, find the distance of the other from the centre

28 How many tangents to a circle can be drawn from a point without it, and how many from a point on the circumference? Can a tangent to a circle be drawn from a point inside it?

29 How many circles can be drawn so as to touch a given straight line at a given point?

30 Find the locus of the centres of all circles which touch a straight line at a given point

31 Show that the locus of the vertices of all triangles upon the same side of the same base and which have the same vertical angle, is the arc of a circle

32 Prove that if one of the sides of a quadrilateral figure inscribed in a circle be produced, the exterior angle is equal to the interior and opposite angle

33 What conditions are essential for describing a circle about a quadrilateral?

34 Reduce the three cases of Proposition 25 to one case

35 Define the angle in a segment of a circle, and the angle of a segment, and show that both these angles with reference to the same circle are together equal to two right angles

36 Show that the locus of the centres of all circles, which pass through two given points, is a straight line

37 Show that if two chords intersect at right angles, the sum of the arcs they intercept is a semicircle.

38. Show that parallel chords of a circle intercept equal arcs.

39. Show that the locus of the vertices of all right-angled triangles, which can be described upon the same hypotenuse, is a circle of which the hypotenuse is a diameter

40 Find the value of the angle in a quadrant.

41. Draw a straight line at right angles to a given straight line from one of its extremities without producing the line.

42 If from any point on the circumference of a circle two chords be drawn at right angles to each other, show that the sum of the squares on these chords is constant Express the *constant* in terms of the radius

43 State the condition when a circle cannot pass through three given points

44 Show that if several circles touch one another internally or externally at a common point of contact, any straight line passing through the point of contact will cut off similar segments from each.

45 In Prop 35 deduce the particular cases from the given proof

46 State the Proposition which is the converse of Proposition (III 37)

47 Show that if from the same point two tangents be drawn to a circle, the tangents are equal

48 Two chords in a circle intersect each other, the segments of the smaller are 12 and 8 inches, and one of the segments of the other is 16 inches, find the length of the remaining segment

49 In figure 1, Proposition 36, DC is equal to 16 inches and DB to 24 inches, find the radius of the circle

50 State the Proposition which is the converse of the Corollary to Proposition 36

51 From a point without a circle two secants are drawn, one of which passes through the centre, the part of the smaller secant intercepted by the circumference of the circle is equal to the radius of the circle, and the parts of the secants outside the circle are 12 and 8 inches in length, find the diameter of the circle.

ADDITIONAL PROPOSITIONS, BOOK III.

Proposition I Theorem. *(Brahmagupta's).*

If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from their point of intersection on any side, being produced, bisects the opposite side

Let the diagonals AC BD, of the quadrilateral ABCD inscribed in the circle ABCD, cut each other at E at right angles.

Let EF be perp. to AB, let FE produced meet DC at G

Then DC *is bisected at* G

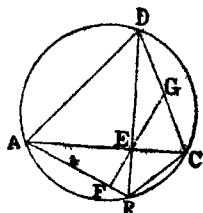
The ∠ DEG = the ∠ BEF
= complement of EBF
= ∠ EAB
= ∠ CDB [III 21.

∴ DG = GE [I 6.

Similarly, CG = GE

∴ DG = CG

Wherefore, *if the diagonals &c* Q E D

Loci.

Proposition II Theorem

The locus of the centres of any number of circles, which touch two intersecting straight lines, are the bisectors of the angles between these straight lines

(See Figure Prop XIII page 98)

Since EF and GP are the locus of a point whose perp. distances from AB and CD are equal,

∴ if from any point in EF or GP as centre, and with radius equal to its perp distance from either of the intersecting lines, a circle be drawn, it will touch both AB and CD, [III 16

∴ EF and GP, the bisectors of the angles between the intersecting lines AB and CD, are the locus of the centres of ⊙s which touch both AB and CD. Q. E. D.

Proposition III Problem

Given the base and the vertical angle of a triangle, to find the locus of the vertex.

Let AB be the given base and C the vertical angle

On AB describe the segment ADB, containing an angle equal to the ∠ C

Then the arc ADB *is the required locus*

Take any point D in the arc ADB, join AD, BD

The △ ADB is on the given base AB and its vertical angle is equal to the ∠ C

If any other point be taken in the arc ADB, the △ formed by joining this point with the extremities of the base AB satisfies the given condition

No point outside the arc is the vertex of the required △ If possible, let the point E be the vertex Join EA cutting the arc at F Join FB

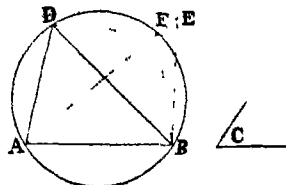
The ∠ AFB is greater than the ∠ AEB [I 16.

But the ∠ AFB = the ∠ ADB, [III 21

= the ∠ C = ∠ AEB,

which is impossible

∴ the point E cannot be the vertex

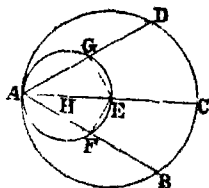
Wherefore, *the arc* ADB *is the required locus* Q. E F

Proposition IV Theorem

If from a given point on the circumference of a circle, any number of chords be drawn the locus of the middle points of the chords will be a circle whose diameter is half of the diameter of the original circle

Let A be the given point on the ○ce of the circle ABCD Find E the centre Draw AC the diameter Bisect AE at H With H as centre and AH or EH as radius describe the ⊙AGF Draw any two chords AB, AD, cutting the smaller ⊙ at F and G, respectively Join FE, GE

AFE is a semi-circle, ∴ AFE is a right angle [III 31

∴ AF = FB [III. 3.

Likewise AG = GD.

Similarly, the middle points of all chords of the larger circle lie on the ⊙ce of the smaller circle.

∴ the ⊙ AGF is the locus of the middle points of all chords of the ⊙ ABCD drawn from A, and its diameter AE is half of AC.

Wherefore, *if from a given point* &c. Q. E. D.

NOTE. If the point were within the ⊙, for instance any point P, and the diameter through P were AC then the required locus is the circumference of a circle whose diameter $= \frac{1}{2}$ AC, whose centre lies in AC, and whose circumference cuts AC at the middle points of PA and PC.

Similarly if the point were without the ⊙, for instance P′, and the diameter through P′ were AC then the required locus is the circumference of the circle whose diameter $= \frac{1}{2}$AC, whose centre lies in P′AC, and whose circumference cuts P′AC at the middle points of P′A and P′C.

Proposition V. Problem

To find the locus of the vertices of all triangles on a fixed base, the sum of the squares on whose sides is constant.

Let AB be the given base, let AEB be a △ so that the sum of the sqs on the sides AE, BE = the sq on C.

Bisect AB at D. Join DE.

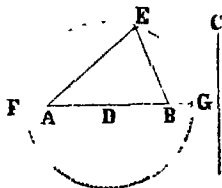
With D as centre and DE as radius describe the ⊙ EFG, *the circle* EFG *is the required locus*.

$$2AD^2 + 2DE^2 = AE^2 + BE^2$$
$$= C^2$$
$$= \text{constant}$$

But AD is constant,

∴ also DE is constant,

∴ the ⊙ EFG is the required locus. Q. E. F.

OBS. Produce, AB both ways to meet the circle in F and G; the sides lie in a straight line when the vertices lie at F or G, and no triangle will be formed the conditions however remain the same. For $AG^2 + BG^2$ or $FB^2 + AF^2 = 2AD^2 + 2DG^2$ [II. 10.

Tangency.

Proposition VI. Problem.

To draw a common tangent to two circles.

Let A be the centre of the greater circle, and B that of the less.

With A as centre, and radius equal to the difference of the radii of the circles, describe a circle From B draw BC touching this circle at C Join AC, and produce it to meet the circle at D

From B draw BE at right angles to CB meeting the ⊙ at E Join DE

DE *is the required tangent*

Because the ∠ ACB is a right angle, [III 18.

∴ DCB is a right angle

But CBE is also a right angle, and DC=EB, for AC=the difference of AD BE,

∴ DCBE is a rectangle [I 29, 34.

∴ DE is at right angles to AD and BE

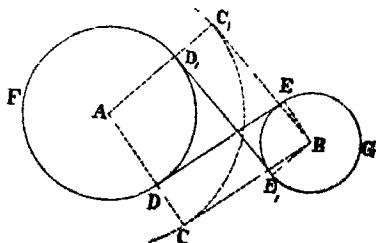
∴ DE touches both the circles [III. 16.

Wherefore, *a common tangent is drawn to both the circles*

Q E. F.

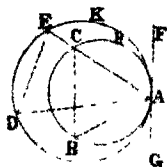
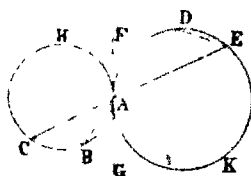
Def The tangent DE is called a ***direct common tangent.***

Obs Another direct common tangent D_1E_1 may be drawn

Note --If the circle be described with A or B as centre and radius equal to the sum of the radii, and the remaining constructions be completed in a similar manner, the common tangent DE shall have each circle on either side of it.

Def. **This tangent is called a *transverse common tangent.***

Obs. D_1E_1 is another transverse common tangent.

Proposition VII Theorem.

If two circles touch each other, internally or externally, the straight line drawn through the point of contact, cutting the circles, will cut off similar segments.

Let the ⊙ ABC touch the ⊙ ADE at the point A

Any line drawn through A cutting the circles will cut off similar segments

Draw any line AC in the ⊙ ABC, and produce it to meet the other ⊙, at E

Because the straight line joining the centres passes through the point of contact [III 11 and 12.

∴ the tangent to any of the ⊙s at A will be tangent to the other

Draw the common tangent FAG

In Fig 1,

the ∠ FAC=the ∠ B in the alternate segment of the ⊙ ABC,
also,
the ∠ FAE = the ∠ D in the alternate segment of the ⊙ ADE.

∴ the segment ABC is similar to the segment ADE

In Fig 2,

the ∠ FAC=the ∠ B in the alternate segment of the ⊙ ABC,
also,
the ∠ EAG=the ∠ D in the alternate segment of the ⊙ ADE.

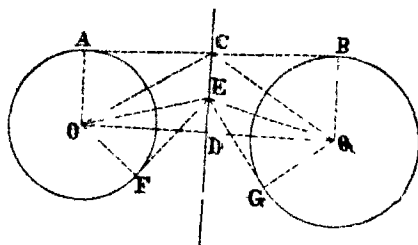
But the ∠ FAC=the ∠ EAG

∴ the segment ABC is similar to the segment ADE.

The other segments are also similar, for they contain angles supplementary to the angles in the former segments. [III 22.

Wherefore, *if two circles touch* &c. Q. E. D

Proposition VIII Problem

To find the locus of points from which tangents to two given circles are equal

Let O and Q be the centres of the two circles Join OQ.
Draw a common tangent AB to touch the circles at A, B.
Bisect AB at C Draw CD perpendicular to OQ
The straight line CD *produced both ways is the required locus.*

Let the circle whose centre is Q, be the greater
Take any point E in CD Draw tangents EF, and EG
Join EO, OF, EQ, QG

$$\because AC^2 = CB^2,$$

$$\therefore OC^2 - OA^2 = QC^2 - QB^2, \quad [I\ 47$$

$$\therefore OD^2 + CD^2 - OA^2 = QD^2 + CD^2 - QB^2, \quad [I\ 47$$

$$\therefore OD^2 - OF^2 = QD^2 - QG^2,$$

$$\therefore OD^2 + ED^2 - OF^2 = QD^2 + ED^2 - QG^2,$$

$$\therefore OE^2 - OF^2 = QE^2 - QG^2, \quad [I\ 47.$$

$$\therefore EF^2 = EG^2, \quad [I\ 47.$$

$$\therefore EF = EG$$

Likewise we can prove that tangents to the two circles from any other point in CD or CD produced are equal

∴ CD is the required locus Q. E. F.

Def 1. *If from every point in a straight line tangents to two circles are equal, that straight line is called the* **radical axis** *of the circles Thus* CD *is the radical axis of the two circles*

NOTE Since $OD^2 - OF^2 = QD^2 - QG^2$, $\therefore OD^2 - QD^2 = OF^2 - QG^2$.

Hence we see that the radical axis cuts the line of centres at right angles, so that the difference of the squares on the two segments of the line, is equal to the difference of the squares on the two radii

If the two circles touch one another, internally or externally, then their common tangent at the point of contact is the radical axis

If the two circles intersect, then the straight line through the points of intersection is the radical axis (III 36)

Beginners may omit the rest of this Note

From the above facts we may infer some very interesting results. For if the two intersecting circles begin to separate, then their points of intersection come nearer to each other, till at last when the two points are consecutive the circles touch one another, and the common secant (i e the radical axis) becomes a common tangent. Now if the circles separate altogether, they do not intersect, hence the common secant through their points of intersection disappears But they still have a radical axis, hence we may say that the radical axis in this case is the common secant which passes through the imaginary points of intersection of the two circles* If we adopt this interpretation, we may define the radical axis of two circles as *the common secant which passes through the two points of intersection (real or imaginary) of the circles*

If one of the circles lie wholly within the other, then, as before, the radical axis cuts *externally*, at right angles the line of centres so that the difference of the squares on the two segments of the line is equal to the difference of the squares on the two radii From this again we may infer a most interesting fact For, since the two radii are constant the difference of the squares on them is constant, therefore the difference of the squares on the two segments of the line of centres is constant, therefore the rectangle contained by the sum and difference of these segments is constant Therefore when the difference of the segments becomes less and less (i e. when the centres come nearer and nearer to each other), the sum of the segments becomes greater and greater (i e the point where the radical axis cuts the line of centres goes further and further) Hence, when the difference of the segments becomes nothing (i e. when the circles become concentric), the sum of the segments becomes infinite, i e. the radical axis cuts the line of centres at an infinite distance But when two circles are concentric every straight line through their common centre is the line of centres. Therefore the radical axis is a line at an infinite distance

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* The student will learn hereafter in the more advanced branches of Mathematics that a real straight line may exist which passes through imaginary points.

on *all sides* of the concentric circles, that is, the radical axis becomes an infinite circle. The way it happens to be so is this :—

Just before the centres of the two circles are about to coincide, the radical axis recedes rapidly, *at the same time spreading its arms around the two circles at an infinite distance*, till when the centres coincide, its arms also completely embrace from an infinite distance the two circles.*

If there be two equal circles then obviously their radical axis bisects at right angles the line between the centres. Hence when the centres coincide, the radical axis, being half way between the centres, must pass through the common centre and *be still at right angles to the line of centres*. But for concentric circles every straight line through the common centre is the line of centres, therefore the radical axis passes through the common centre and is at right angles to every line through it; that is, the radical axis in this case is any straight line through the common centre. And since a straight line may be drawn to the centre from any point whatever, we obtain the curious result that *when the centres of two equal circles coincide, their radical axis becomes the whole plane of the common circles*. This is obvious, since when the centres coincide the circumferences also coincide, and from any (external) point on the plane the two tangents are equal.

Obs. One fixed straight line may be the radical axis of any number of pairs of circles, and in such a case the circles are said to be ***co-axal***.

Def.* 2.** *The point from which tangents to three given circles, whose centres are not in a straight line, are equal is called the* ***radical centre *of the three circles.*

NOTE. That such a point exists is seen from the Prop. that the three radical axes of three circles (whose centres are not collinear) taken two and two are concurrent :—

For if A, B, C, be three such ⊙s,
and the radical axis of ⊙ A and ⊙ B meet the radical axis of ⊙ B and ⊙ C at the point P,
then the tangent from P to ⊙ A = the tangent from P to ⊙ B,
but the tangent from P to ⊙ B = the tangent from P to ⊙ C
∴ the tangent from P to ⊙ A = the tangent from P to ⊙ C,
∴ the radical axis of ⊙ A and ⊙ C also passes through P.

Def.* 3.** *Any radius of a circle is a* ***normal *to it.*

Def.* 4.** *Two circles are said to cut one another* ***orthogonally *when the tangent to one at any point of intersection is a normal to the other.*

---

* From this investigation we may deduce the definition of a **line at infinity** :—a line at an infinite distance which embraces completely the plane in question, like the horizon, *but any finite portion of which is a straight line*.

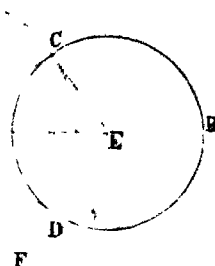
Proposition IX. Problem.

To describe a circle with a given centre to cut a given circle orthogonally

Let A be the given centre of the circle to be described, and CBD be the given circle

Find E the centre of the given circle Join AE Draw AC and AD, tangents to the ⊙ BCD Join CE, DE

AC = AD

With A as centre and AC or AD as radius describe the ⊙ CDF

The circle CDF cuts the circle CBD orthogonally.

The ∠s ACE and ADE are right angles [III 18

∴ CE and DE touch the ⊙ CDF [III 16

∴ CE and DE, normals to the ⊙ CBD, are tangents to the ⊙ CDF

Also, AC and AD normals to CDF are tangents to the ⊙ CBD

Wherefore *the circle CDF cuts the circle CBD orthogonally*

Q E F.

Proposition X Theorem

The radical axis of two circles is the locus of the centres of circles which cut the two circles orthogonally

Draw CD the radical axis of the two given circles whose centres are O, Q [See Fig Prop VIII

CD is *the locus of the centres of circles which cut the given circles orthogonally*

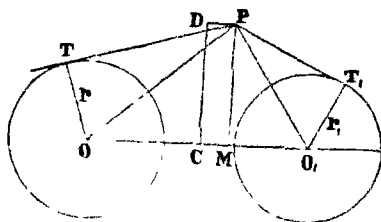
Take any point E in CD. Draw EF, EG, tangents to the ⊙s whose centres are O and Q, respectively.

Then EF = EG [Prop VIII.

∴ a circle described with E as centre, and EF or EG as radius, will cut the given ⊙s orthogonally. [Prop. IX.

Wherefore, *the radical axis of two circles &c.* Q. E. D.

Proposition XI. Theorem

If from any point tangents be drawn to two circles, the difference of the squares on them is equal to twice the rectangle contained by the perpendicular let fall from the point on the radical axis and the distance between their centres.

Let O and O_1 be the centres of any two ⊙s, and CD their radical axis. Let P be the point from which tangents are drawn to the two ⊙s, and PD the perpendicular on the radical axis. It is required to prove that $PT^2 \sim PT_1^2 = 2PD \cdot OO_1$

$$\because \ PT^2 = PO^2 - r^2, \text{ and } PT_1^2 = PO_1^2 - r_1^2$$

$$\therefore \ PT^2 \sim PT_1^2 = PO^2 \sim PO_1^2 - (r^2 \sim r_1^2)$$
$$= (OM^2 \sim O_1M^2) - (OC^2 \sim O_1C^2)$$
$$= (OM + O_1M)(OM \sim O_1M) - (OC + O_1C)(OC \sim O_1C)$$
$$= OO_1\{(OM \sim O_1M) - (OC \sim O_1C)\}$$
$$= OO_1\{(OM - OC) + (O_1C - O_1M)\}$$
$$= OO_1 \cdot 2CM = 2PD \cdot OO_1$$

Wherefore, *if from any point* &c. Q. E. D.

MISCELLANEOUS EXERCISES ON BOOK III.

Exercises Solved

Proposition 1. Problem

To construct a square equal to the difference of two squares.

On the side of the greater square as diameter, describe a semicircle

With one extremity of this diameter as centre, and with radius equal to the side of the smaller square, describe a ⊙

Then the square on the line joining the point of intersection of the two ⊙ces with the other extremity of the diameter

=the difference of the given squares

since the ∠ in a semicircle is a right angle Q E F

Proposition 2 Problem

If a square formed by four equal rods hinged at the corners, be placed vertically so that one side lies along the ground, and while this side remains fixed the figure be swung in the same vertical plane, to find the locus of the intersection of the diagonals

When the figure is swung, it becomes a rhombus, for the sides still remain equal but the angles become unequal.

But the diagonals of a square as well as of a rhombus intersect at right angles

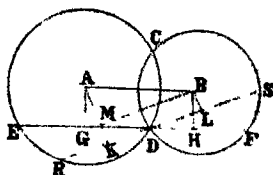
∴ the locus of the intersection of the diagonals is a semicircle having the fixed side as diameter,

since the ∠ in a semicircle is a right angle. Q. E. F.

Proposition 3. Problem

Two circles cut each other; to draw through one point of intersection a line the sum of the segments of which intercepted by the circles shall be the greatest possible

Let A and B be the centres of two ⊙s which cut each other at C, D

Join AB, and draw EDF ∥ AB. *Then* EDF *is the required line.*

Draw any other line RDS

Draw AG, BH, ⊥s to EF, and AK, BL, ⊥s to RS. Draw BM ∥ RS

Then BM = KL

∵ ED is double of GD, and DF double of DH, [III. 3.

∴ EF is double of GH

Similarly RS is double of KL or BM.

But KL or BM is less than AB, that is, less than GH,

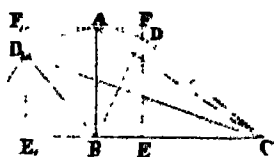
∴ RS is less than EF.

∴ *of all lines drawn through* D, EF *is the greatest* Q. E. F.

Proposition 4 Problem

Two sides of a triangle are given, to construct a triangle of maximum area

Let AB, BC, be the given sides. it is required to find at what ∠ they must be placed so that the △ ABC be the maximum.

Now let BC remain fixed, and the side BA make different ∠s with BC.

Then whatever different positions BA may take, the point A will always lie on the Oce of the circle described with centre B and radius BA

Then, since BC remains fixed, the △ formed has the maximum area when the altitude of A on BC (or BC produced) is the greatest, that is, when AB is ⊥ BC

Hence the maximum △ is obtained when the two sides are at right angles to one another Q. E. F.

Proposition 5. Theorem.

Of all rectangles of a constant area, the square has the minimum perimeter.

Let any number of chords intersect at a point E within a ⊙ ABCD

Then the rectangles contained by the segments of each chord are all equal [III 35

Hence, whatever position a chord passing through E might occupy, the rectangle contained by its segments is constant in area

But of all such chords, that which is bisected at E is the shortest, [III 15

and the rectangle corresponding to it becomes the square on half the chord

Moreover the perimeter in each case = twice the chord (*i e* twice the sum of *length* and *breadth*)

∴ the shortest chord gives the least perimeter of the corresponding rectangle

∴ the square has the least perimeter Q E D

NOTE This Prop may be stated algebraically, thus -

$$\text{If } ab = \text{constant}$$
$$\text{then } a+b = \text{minimum,}$$
$$\text{when } a=b$$

OBS The above investigation enables us to obtain some practical results

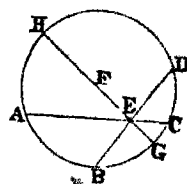
(1) If we had to construct a rectangular court yard (or to enclose a rectangular field) of a given area so that the cost of making a road around it was to be the least, we should make the court yard (or the field) a square

(2) If we had to dig a reservoir with vertical sides and of a fixed depth, to contain a given quantity of water so that the cost of lining it with masonry was to be the least, we should make it in such a manner that the surface would be a square thus —

Let the quantity of water the tank must hold $=k$ cubic ft

and let the fixed depth of the tank $=d$ ft.

$$\text{Then } \frac{k}{d} \text{ sq ft} = \text{the area of the surface}$$

$$\therefore \sqrt{\frac{k}{d}} \text{ ft.} = \text{each side of the surface.}$$

MISCELLANEOUS EXERCISES ON BOOK III.

1. The straight line which bisects two parallel chords in a circle is also perpendicular to them.

2. Perpendiculars from the extremities of a diameter of a circle upon any chord, cut off equal segments.

3. Two circles cut one another, and through the points of intersection, two parallel straight lines are drawn which are terminated by the circumferences of both the circles; shew that they are equal.

4. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

5. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first, through the point where it cuts the less circle. Show that these two chords have the segments of the one equal to the segments of the other each to each.

6. Two straight lines make an angle between them; describe a circle which shall have a given radius, and its centre in one of the lines, and shall touch the other.

---

7. In a given circle draw a chord which shall be equal to a given chord and shall be bisected by another given chord greater than the former.

8. From a given point as centre describe two circles each of them touching a given circle.

9. A circle touches another internally; describe a third circle touching the smaller circle externally and the larger internally.

10. Describe a circle about a given oblong.

11. Inscribe a circle in a given sector of a circle.

12. Describe a circle which shall have a given radius, shall have its centre in a given straight line, and shall also touch a given circle.

13. Describe a circle which shall have a given radius, and touch a given circle and a given straight line.

14. Draw a common tangent to two given circles.

15. Describe a circle which shall pass through a given point, and touch a given straight line at a given point in the same.

16. Draw a straight line cutting two given circles, so that the chords intercepted within the circles, shall be equal to two given chords, one in each circle.

17. Describe a circle which shall touch a given circle, and also touch a given straight line at a given point.

18. With the three angular points of a triangle as centres describe three circles, so that each shall touch the other two.

19 Describe two circles, each having a given radius which shall touch a given straight line, not at the same point, and shall also touch one another

20 AB, CD are parallel diameters of two circles, and AC cuts *the circles in P, Q, prove that the tangents to the circles at P, Q* are parallel

21 SR and TN are two equal circles, of which the centres are O, P respectively, not touching or cutting each other Join their centres O and P It is required to draw a straight line touching the circle TN which being produced shall cut the line *OP*, and touch the circle SR

22 Describe a circle with a given centre, bisecting the circumference of a given circle

23 Find a point without a given circle, from which, if two straight lines be drawn touching the circle they shall form an equilateral triangle with the chord which joins the points of contact

24 If two circles touch each other externally and on the part of their common tangent intercepted between the points of contact, as diameter, a circle be described, it will touch the line joining the centres

25 If on the diameter of a circle as base a right angled triangle be described, the tangent drawn from the point in which the hypotenuse cuts the circumference of the circle will bisect the perpendicular

26 If two circles touch each other externally the square on the common tangent is equal to the rectangle contained by the diameters of the circles

27 In the diameter of a circle produced determine a point so that the tangent drawn from it to the circumference, shall be of given length

28 If the diameter of a circle be divided into two unequal parts and two other circles be described upon these parts as diameters, prove that the square described on the part of the common tangent to the two circles intercepted between the points of contact shall be equal to the rectangle contained by the parts of the diameter of the original circle

29 Show that all equal straight lines in a circle will be touched by another circle

30 Two points on the circumference of a circle are given and the middle point of the arc is taken, describe two equal circles which shall touch one another in the middle point, and pass, one through one of the extreme points, and the other through the other

31 Given a circle and a straight line find a point in the straight line so that the tangent to the circle drawn from the point may be equal to another given straight line

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32. If two circles touch each other externally, and a common tangent be drawn touching them both on the same side, the tangent subtends a right angle at the point of contact

33 If two chords of a circle cut one another, the angle between them is half the sum or difference of the angles subtended at the centre by the arcs intercepted between them, according as they cut one another within or without the circle

34. Of all triangles upon the same base and between the same parallels, the isosceles triangle has the greatest vertical angle

35 If two straight lines AEB and CED in a circle, intersect at E, the angles subtended by AC and DB at the centre, are together double of the angle AEC

36 If two straight lines, whose extremities are in the circumference of a circle, cut one another, the triangles formed by joining their extremities are equiangular to each other

37 An infinite number of triangles having equal vertical angles are described on a given base, show that the straight lines bisecting the vertical angles pass through a fixed point

38 If the chords, which bisect two angles of a triangle inscribed in a circle be equal prove that either the angles are equal or the third angle is equal to the angle of an equilateral triangle

39 APB and AQB are any two angles in the segment of a circle subtended by the straight line AB AP and BQ produced, meet at R, also AQ and BP intersect at S Prove that, if a circle can be described about the quadrilateral PRQS, the line AB is the diameter of the circle APQB

40 If two circles cut one another, and from any point in the circumference of the one straight lines be drawn through the points of intersection to meet the other, the angle contained in the segment which they intercept, is always the same, whatever point be taken in whichever circle

41 From the centre of a circle two straight lines are drawn containing a right angle and meeting a given straight line at A, B, and from A, B tangents are drawn so as to form a quadrilateral circumscribing the circle Show that this quadrilateral can be inscribed in a circle

42 Let two circles cut each other at the points A, B From A draw two straight lines ACD, AEF on the same side of AB and cutting both the circles, make the two triangles CEB and DBF, and prove that they are equiangular

43 AB, CD are two chords cutting each other at right angles at the point E in the circle ACBD BF is the tangent at B and DF is the straight line drawn from D at right angles to BF Show that the triangles ABD and DEF are equiangular

44. Show that a circle cannot be described about a rhombus.

45 Show that no parallelogram except a rectangle can be inscribed in a circle.

46 Divide a circle into two segments, so that the angle contained in one segment is equal to three times the angle in the other

47 Two circles intersect at A, B, PAR, QAS are drawn equally inclined to AB to meet the circles at P, R, Q, S, prove that PR is equal to QS.

48. Draw two tangents to a given circle which shall contain an angle equal to a given rectilineal angle

49 Draw two concentric circles, such that those chords of the outer circle, which touch the inner, may be equal to its diameter.

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50. If two circles touch each other, and if two straight lines be drawn from the point of contact to cut the circles, the chords joining the points of intersections are parallel

51 If one circle touch another internally at P and a line ABCD be drawn cutting the circles at the points A, B, C, D, then the angle APB is equal to the angle CPD

52 If any chord of a circle be produced, until the part produced be equal to the radius and its extremity be joined with the centre of the circle, and produced to the circumference of the two arcs intercepted between these two straight lines the one is three times the other

53 A chord of a circle is the base of an isosceles triangle whose vertex is without the circle and whose equal sides cut the circle Prove that the triangle formed with the straight line joining the points of intersection, is isosceles

54 The straight lines bisecting any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle

55 Two equal circles cut one another at the points A, B; BC is a chord equal to AB show that AC touches the other circle

56 If a quadrilateral ABCD be inscribed in a circle and AB DC be produced to meet at E, then show that the triangles EBC EAD will be equiangular

57 If through a point in the circumference of a circle two chords be drawn and the arcs which they cut off be bisected, the straight line which joins the points of bisection shall cut off equal portions of the chords measured from the given point

58 If A B C be three points in a straight line and D a fixed point at which AB BC subtend equal angles, show that DB passes through another fixed point

59 If two chords of a circle intersect at right angles, the portions of the circumference taken alternately are together equal to half the circumference

60 If two equal chords of a circle cut one another, the segments of the one shall be equal to the segments of the other, each to each

61 The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point

62 Two equal circles intersect at A, B, PTQ perpendicular to AB meets it at T, and the circles in P, Q, AP BQ meet at R; AQ BP at S, prove that the angle RTS is bisected by TP

63 A triangle is turned about its vertex, until one of the sides intersecting in that vertex, is in the same straight line as the other

previously was, prove that the line, joining the vertex with the point of intersection of the two positions of the base, produced if necessary, bisects the angle between these two positions.

64. *If two equal circles cut one another, and through one of the points of intersection, a straight line be drawn cutting them both, the points of section are equidistant from the other point of intersection of the circles.*

65. In a circle, a quadrilateral figure is inscribed, two opposite sides of which are equally distant from the centre but are not parallel, show that the other two sides are parallel.

66. If any number of triangles upon the same base BC, and on the same side of it, have their vertical angles equal, and perpendiculars meeting at D be drawn from B, C upon the opposite sides, find the locus of D and show that all the lines which bisect the angle BDC, pass through the same point.

67. If from any point in the diameter of a semi-circle, two straight lines be drawn one to the middle point of the circumference, and the other at right angles to the diameter meeting the circumference, show that the squares on those straight lines, are together double the square on the radius.

68. If a straight line be drawn from a point in the diameter produced so that the part of the secant without the circle be equal to the radius, show that the concave arc is three times the convex arc.

69. If two tangents be drawn from a point without a circle, the angle contained by the tangents is double of the angle contained by the line joining the points of contact, and the diameter drawn *through one of them.*

70. The greatest rectangle that can be inscribed in a circle is a square.

71. If from any point in the circumference of a circle, a chord and a tangent be drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal to one another.

72. Through a given point within a given circle, to draw a chord so that the parts of it between the point and the circumference, shall have a given difference.

73. If two circles cut one another and from one of the points of intersection two diameters be drawn, their other extremities and the other point of contact will be in one straight line.

74. Two equal circles touch each other externally, and through the point of contact chords are drawn, one in each circle, at right angles to each other, prove that the straight line, joining the other extremities of these chords, is equal and parallel to the straight line joining the centres of the circles.

75. The vertical angle of any oblique-angled triangle inscribed in a circle is greater or less than a right angle, by the angle contained by the base, and the diameter drawn from the extremity of the base.

76. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

77 If from the extremities of any diameter of a given circle perpendiculars be drawn to any chord of the circle, produced if necessary, the less perpendicular shall be equal to the segment of the greater, contained between the circumference and the chord

78 The circles described on two of the sides of a triangle as diameters, intersect on the base or the base produced

79 Draw a chord in a given circle which shall subtend a right angle at a given point within the circle, and be parallel to a given straight line

80 SMPR is a semicircle on SR Let the chord SP and RM cut each other at N Prove that the square on SR is equal to the rectangle contained by SN and SP, together with the rectangle contained by RN and RM

81 If two tangents to a circle cut one another the straight line drawn from the centre to the point of section is parallel to the straight line drawn from one point of contact to the extremity of the diameter which passes through the other

82 If two circles cut one another, and from either point of intersection diameters be drawn, so that each of them shall touch the other circle, the rectangle contained by the straight lines joining the extremities of these diameters and the other point of intersection shall be equal to the square on the line joining the points of intersection

83 CA is the radius of a circle of which the centre is C, B a point in CA, find the point of the circumference at which CB subtends the greatest angle

84. If two chords of a circle intersect each other at right angles either within or without the circle the sum of the squares described upon the four segments is equal to the square described on the diameter

85 Find the locus of the middle points of all chords of a circle, which pass through a fixed point in the circumference of the circle

86 From one extremity of the diameter of a circle, to draw a straight line to the tangent at the other extremity, which shall be bisected at the circumference of the circle

87 Draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater, may be twice the part intercepted by the circumference of the less

88. ACDB is a semicircle on AB as diameter On CD a chord produced towards D, a perpendicular BE is drawn Show that the square on AB is equal to the squares on AC, CD, DB together with twice the rectangle contained by CD, DE

89 The circles described on the two sides of any triangle as diameters, will intersect in the remaining side, produced if necessary

90. ABCD is a square and E any point in BC; EF is drawn perpendicular to AE, meeting in F the line CF, which bisects the angle between CD and BC produced: prove that AE is equal to EF.

91 Describe a circle with a given radius touching a given line, such that the tangents drawn to it from two given points in

the straight line (the distance between the points being not less than the diameter), may be parallel

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92. If two circles touch each other, and parallel diameters be drawn, the straight lines which join the extremities of these diameters will pass through the point of contact

93 If two circles touch one another internally or externally, and a line cutting both be drawn through the point of contact, the arcs cut off, shall subtend equal angles at the centres of their respective circles

94 If two circles touch each other, any straight line drawn through the point of contact, will cut off similar segments

95 AB CD are parallel diameters of two circles, and AC cuts the circles at P, Q prove that the tangents to the circle at P, Q, are parallel

96 The opposite sides of a quadrilateral inscribed in a circle, are produced to meet at P, Q, and about the four triangles thus formed, circles are described prove that the tangents to these circles at P and Q, form a quadrilateral equal in all respects to the original, and that the line joining the centres of the circles about the two quadrilaterals, bisects PQ

97 If from one extremity of a chord of a circle there be drawn a tangent, and a perpendicular to the diameter which passes through the other extremity, the angle between these two lines will be bisected by the chord

98 The chord which joins the points of contact of parallel tangents is a diameter

99 ABC, ADC are two triangles on the same base AC, and having equal vertical angles If AD BC intersect in E, show that the rectangle AE ED is equal to the rectangle BE EC

100 An acute angled triangle is inscribed in a circle, and the paper is folded along each of the sides of the triangle show that the circumferences of the three segments will pass through the same point

101 If a chord to a circle be produced equally both ways, and from the extremities, tangents be drawn to the circle on opposite sides of the chord, the straight line which joins the points of contact shall bisect the chord

102 If from any point in the circumference of a circle, a chord and a tangent be drawn, the perpendiculars dropped on them from the middle point of the subtended arc, are equal to one another.

103. Two circles intersect in A and B At A, the tangents AC, AD are drawn to each circle and terminated by the circumference of the other If BC, BD be joined, show that AB or AB produced, will bisect the angle CBD.

104. Two equal circles touch one another externally at A, and a straight line BAC is drawn terminated by the circles at B and C. Show that the centre of the circle which passes through C and touches the circle AB at B, lies on the circumference of the circle AB.

105 Construct a triangle of which the base, the sum of the other two sides, and the vertical angle are given.

106 Construct a triangle of which the base, the angle opposite the base, and the altitude are given

107 Given the base, the difference of the other two sides, and the angle opposite to the base, construct the triangle

108 Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls

109 Draw a common tangent to two circles in a transverse direction, so that the circles may lie on opposite sides of the tangent

110 Given the perimeter, the altitude, and the vertical angle of a triangle, to construct the triangle

111 Given the base, the vertical angle, and the median that bisects the base, construct the triangle

112 Given the base, the vertical angle and the length of the straight line drawn from the vertex to the middle point of the base, to construct the triangle

113 Given a side of a triangle, its vertical angle and the radius of the circumscribing circle, to construct the triangle

114 Divide a circle into two segments such that the angle in one of them shall be five times the angle in the other

115 Through a given point without a circle draw a chord, such that the difference of the angles in the two segments, into which it divides the circle may be equal to a given angle

116 To find a point from which three straight lines drawn to three given points, not in a straight line shall make equal angles with each other

117 The angle contained by the tangents drawn at the extremities of any chord in a circle is equal to the difference of the angles in the segments made by the chord

118 If straight lines be drawn from a fixed point to the circumference of a circle the locus of their middle points is a circle

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119 Given the segments of the base (made by a straight line bisecting the vertical angle) and the vertical angle to construct the triangle

120 If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle

121 Through a point within a circle, draw a chord, such that the rectangle contained by the whole chord and one part, may be equal to a given square

122. AB and AC are tangents to the circle CFB; at whatever point between C and B, the tangent EFD is drawn, the three sides of the triangle AEID are equal to twice AB or AC, also the angle subtended by the tangent EFD at the centre of the circle, is constant.

123. If a line be drawn from the centre of the inscribed circle perpendicular to the base of the triangle, then either of the sides of the triangle and the opposite or remote segment of the base is equal to half the perimeter of the triangle.

124. If a circle be described touching the base of the triangle and the two sides produced, the line intercepted between the point of contact and the vertex of the triangle, is equal to the semi-perimeter.

125. BA, AC are two chords in the same straight line of two circles which intersect at A. From B, a tangent BD is drawn to the circle ADC, and from C a tangent CE to the circle BEA. With centres B and C at the distances BD, CE respectively, circles are described intersecting at F. Join BF, FC and show that the angle BFC is a right angle.

126. If two chords AB, AC be drawn from any point A in the circumference of a circle, and be produced to D and E, so that the rectangle AC, AE is equal to the rectangle AB, AD, then if O be the centre of the circle, AO is perpendicular to DE.

127. If from a given point A without a given circle, any two straight lines APQ, ARS be drawn making equal angles with the diameter which passes through A, and cutting the circle at P, Q and R, S respectively: then PS, QR shall cut one another at a given point.

128. If two straight lines cut each other, and the rectangle contained by the segments of the one be equal to the rectangle contained by the segments of the other, the ends of the straight lines are concyclic.

129. If two tangents be drawn at the extremities of the diameter of a circle, and a third tangent be drawn at any other point to meet them, the rectangle by its segments between the other two tangents and the point of contact is equal to the square described on the radius.

130. If two circles intersect at A and B, and CBD be drawn perpendicular to AB to meet the circles in C and D, and if EAF bisect either the exterior or interior angle between CA and DA, prove that the tangents of the circles at E and F intersect at a point on AB produced.

131. Two chords AD, BC are drawn in a semicircle from the extremities of the diameter AB, the chords intersect at E, prove that the square on the diameter AB, is equal to the squares on AE, BE, together with twice the rectangle contained by AE, ED.

132. From a given point without a circle whose distance from the circumference is not greater than the diameter, to draw a secant which shall be bisected by the circumference.

133. From a point A without a circle BDCE, two tangents AB, AC, and a secant ADE are drawn, perpendiculars BG, CF are drawn on AE, show that the difference of the squares on BD, DC, is double of the rectangle contained by AD, FG.

134. Given the vertical angle, the difference of the sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; construct the triangle.

135. If two circles touch one another externally, and a common tangent be drawn, not meeting both at the same point, the square

on the part of this line, intercepted between the points of contact, is equal to the rectangle contained by the diameters of the circles.

136 From a given point as centre, describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line, may be equal to a given square

137 ABC is a triangle whose acute vertex is A; show that the square on BC is less than the squares on AB, AC by twice the square on the line which is drawn from A to touch the circle on BC as diameter

138 Describe a circle which shall have its centre in a given straight line, shall pass through a given point and touch another given straight line not parallel to the former

139 From a given point without a circle at a distance not greater than the radius, draw a secant so that the part of it within the circle, may be double the part without it

140 If there be two concentric circles and any chord of the greater be cut by the less at any point the rectangle contained by the two segments into which the point divides the chord, is invariable

141 From any point in a given straight line as centre describe a circle passing through a given point in another given straight line, not parallel to the former so that the tangent from the point of intersection, may be equal to a given straight line

142 The rectangle contained by the sides of a triangle, is equal to the rectangle contained by the perpendicular to the base from the vertex and the diameter of the circumscribing circle

143 Describe a circle which shall touch each of two given straight lines (not parallel) and shall pass through a given point

144 Describe a circle passing through a given point, having its centre on a given straight line and touching a given circle

145 Find a point in the straight line which touches a circle at the end of a given diameter, such that if a straight line be drawn from this point to the other extremity of the diameter the rectangle contained by the part of it without the circle and the part within the circle, may be equal to a given square not greater than that on the diameter

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HINTS FOR SOLUTION.

BOOK III.

Prop. 1.

1 Join the extremities of the arc with any point in the circumference of the arc, bisect these lines at right angles by two straight lines The point at which the latter lines meet, is the centre of the circle

2 Join the given points A, B C, so as to form a triangle Bisect AB at D and BC at E From D, E draw DF, EF, perpendiculars to AB BC respectively F is the centre of the required circle

3 The line is the locus of the point equidistant from the given points

4. The centre of the circle lies in all the bisectors

Prop 2

2. Let the ⊙ pass through two of the extreme points; the other point must be in the line joining the two points

Prop 3

1 Apply Exc I 8 and 4

2. From the common centre draw a perpendicular to the straight line and apply Exc III 3

3 Let A be the centre and B the given point Join AB. Through B draw a chord at right angles to AB

5 The straight line bisecting at right angles the line joining the two points meets the given straight line at a point, which is the centre of the required circle

6 Let AB be the given chord in the circle whose centre is C P the given point APB is a right angle $CE \perp AB$ bisects AB at E. Also EP is half of AB (Addl Prop IV page 91) $\therefore EP$ is constant, ∴ the ⊙ with P as centre and PE as radius is the locus of E

Prop 4.

The point of bisection is the centre.

Prop 5.

The lines bisecting the lines joining the three points at right angles, are concurrent at the centre of each circle. Whence the two circles are concentric and they have equal radii.

Prop 6.

1 Let the circumference of the ⊙ ACB meet that of the ⊙ ABD in the two points A, B. The straight line FG bisecting AB at right angles contains the centre of each circle. But these ⊙s are not concentric. Let E be the centre of ⊙ ACB, and F of circle ADB. Join AE, BE, AF, BF. Produce FG to cut the ⊙ ACB at H and ADB at K. $FH = FE + AE > AF (= FK,)$ which is impossible.

Prop 7

1 Join the point with the centre and produce to meet the circumference.

2 With the point as centre describe a ⊙ to cut the original circle. Bisect the ∠ between the lines joining the points of intersection with the given point.

Prop 12

2. Apply Euc I 27.

Prop 14

1 Euc. I 26.

2 The points of bisection are equidistant from the centre of the circle.

Prop 15

2 See Addl Prop IV page 279.

3 With A any point on the circumference of the circle as centre and radius equal to the given straight line, describe a ⊙ cutting the given circle of centre C, at B. Draw $CD \perp AB$. Let P be the given point.

On CP describe a semicircle CEP. With centre C and radius CD describe a circle cutting CEP at F. Join FP and produce it (both ways if necessary) to meet the circumference of the original circle.

Prop. 16.

2. From C the centre of the given circle draw $CD \perp AB$ the given straight line. Draw $CE \perp CD$ and $EF \perp AB$.

3 Let the straight line joining the centres of the circles cut them at C, D Bisect CD at E. With E as centre and EC or ED as radius describe a circle.

4. Let AB be the given straight line in which the centre shall lie, and AC the line which the circle shall touch Draw AD at right angles to AC and make AD equal to the given radius Draw DE parallel to AC meeting AB at E, draw EF perpendicular to AC. The circle with E as centre and EF as radius is the required circle.

Prop. 17

1 Let AB, AC, be two tangents to a circle Find the centre O Join BO, CO, BC The angle ABC is equal to the angle ACB.

2 Apply Ex 1

Prop 18

3 See Addl Prop II page 278

4. Let P be the given point and Q the given point in AB. Draw $QC \perp AB$ Make the angle $QPC = PQC$

5 Let A be the given point and B the given point on the circumference of the circle whose centre is C Join CB, AB. Produce CB to D and make the angle $BAD = ABD$

6 Let D be the given point in the given line AB, C the centre of the ⊙ Draw DE at right angles to AB on the side remote from the circle and make DE = radius of the given circle. Join CE Make the angle $ECG = CED$, let GC cut the circle at F and meet ED produced at G

The ⊙, with G as centre and GD or GF as radius is the required (⊙)

7 Let P be the given point in the given circle whose centre is C, let PD be $\perp PC$ PD meeting BD the given straight line at D Bisect the angle PDB by DA cutting CP produced at A. Draw $AB \perp BD$ The ⊙ with A as centre and AP or AB as radius is the required circle

8 Let P be the point on the circumference of the circle whose centre is A Join PA and produce it to C making PC = radius of the other ⊙ whose centre is B Join BC At B make the angle $CBE = ACB$, BE meeting CA produced (if necessary) at E The circle with centre E and radius equal to EP is the required one

9 The third line will form a triangle The bisectors of the angles are concurrent (Addl Prop XVI page 101) The point of concurrency is the centre and the perpendicular from the point on any of the lines is the radius of the required circle

10 See Addl. Prop. VI, page 281.

Prop. 19

Let A be the given point Draw any chord BC equal to the given straight line Find the centre O Bisect BC at D From the centre O and radius OD describe a circle Through A draw a tangent to this ⊙, and produce it to cut the given ⊙ at E and F EF is the required chord

Prop 21

2. See Addl Prop I at the end of Book IV

5. Produce BD to E making DE equal to DC, join CE DCE may be proved equilateral (III 21) Hence the angle ACD is equal to the angle BCE

6. Let ABC be a triangle of which the base BC and the vertical angle BAC are given Draw $BD \perp AC$ and $CE \perp BA$, cutting each other at F The angle EFD is the supplement of the angle A, ∴ the angle BFC is known

The locus of F is a segment See Ex 4

7 The fixed point is the middle point of the arc AB

8. The ∠ AOB is constant · hence the locus of O is the arc of a segment (on AB as chord) which contains this angle

Prop 22

1 Addl Prop III, at the end of Book IV.

6. Apply Euc I 32.

7 Divide the polygon into quadrilaterals.

8. Apply Euc I 14

9 Draw any diameter AB At A and on both sides of BA make the angles BAC, BAD, each equal to two thirds of a right angle Join CD

Prop 26

3 Let the two chords AB, CD intersect in O. Join AD, CB Then the ∠ ADC + the ∠ BAD = a right angle ∴ the arc AC + the arc BD subtend a right angle at the circumference

Prop 28

Let AB be the given arc in the given circle ABC Let P be the given point Join AB Find the centre O From O draw OR perpendicular to AB With radius OR describe a concentric circle RHK Draw PE touching the circle RHK at E and cutting the outer circle at F, G

Prop 30.

1. The bisectors of the vertical angles bisect the opposite arc.

2. The external bisectors meet at the other end of the diameter drawn through the middle point of the opposite arc.

Prop 31

3 The line drawn from the point of intersection to the centre of the larger circle is at right angles to the chord

4. Join the given point with the centre of the circle. The circle described with this line as diameter cuts the given circle in two points; lines drawn from these points to the given point are the required tangents.

5 Let AB, CD be two chords which cut each other at right angles at the point G. Let O be the centre. Draw $OE \perp AB$, and $OF \perp CD$.

Join OG. $AG^2+GB^2+CG^2+DG^2$

$=2AE^2+2EG^2+2CF^2+2FG^2$

$=2AO^2+2OC^2$

$=4AO^2$.

Prop 33

1 On the given base describe a segment containing an angle = the vertical angle.

2 On the given base BC describe a segment BAC containing an angle = the vertical angle. Draw $BD \perp BC$ and make BD = the given perp., draw $DA \parallel BC$ cutting the ⌢ at A. BAC is the required △

3 (1) On the given base BC describe the segment BAC containing an angle = the vertical angle, and the segment BDC containing an angle = ½ the angle BAC. In the segment BDC place the line BD = the given sum, cutting the segment BAC at A. BAC is the triangle required.

(2) In the 2nd case the second segment is to contain an angle = the sum of the angles BAC and half of its supplement.

4 On the given base AB describe a segment ADB containing an angle equal to the given vertical angle, also describe a semicircle ACB on AB. In the semicircle draw a chord AC equal to the given perpendicular. Join BC, and let it, produced if required, meet the segment ADB at D. ADB is the required triangle.

Prop 34.

1 Let P be the given point From the given circle ABC cut off the segment ABC, containing an angle equal to the given angle (III 34) From the centre O draw OD perpendicular to AC From the centre O and with radius OD describe the circle EF Draw PE touching the circle EF at E and cutting the circle ABC at G, H PH is the required line,

2 Describe a circle ABC with the given radius Cut off a segment ACB containing an angle = the given vertical angle Make the angle BAC = the given angle at the base ABC is the required triangle

Prop 35

2 & 3 Apply Euc II 5

5 Apply Ex 4

Prop 37

1 Each tangent is equal to the rectangle contained by the common chord produced and the part produced

The point in which the tangents meet is the point where the lines bisecting the angles of the triangle formed by joining the centres also meet

3 Let the straight line joining the two given points A, B, produced, meet the given straight line CD at C Describe any circle ABK passing through A B Draw tangent CK From CD cut off $CP = CK$ Draw perpendiculars from the middle points of AB, AP and let them meet at O O is the centre of the required circle whose radius is PO or OA

Miscellaneous Exercises on Book III

1 Apply Euc III 3

2 From the centre draw a perpendicular to the chord also draw through the centre a straight line parallel to the chord.

3 From the centres draw perpendiculars to one of the chords, produce them to meet the other, and apply Euc III 3

4 Join the centres A B and through one of the points of intersection draw a straight line parallel to AB.

5 From the centre of the larger circle draw a line perpendicular to the second chord and prove that it is equal to the radius of the smaller circle Apply Euc III 14.

6. Let AB, BC be the two given straight lines. From A draw AD at right angles to AB making it equal to the given radius. Through D draw DE parallel to AB, meeting BC at E. E is the centre of the required circle. (Euc. III. 16.)

7. Of the two chords AB, CD, let AB be greater than CD. Find the centre E, and from E draw EF perpendicular to CD. With EF as radius, describe a concentric circle cutting AB at G. Join EG. Through G draw a chord at right angles to EG.

8. *Join the given point with the centre of the given circle* and produce it to meet the circle again. The circles are to pass through the points of intersection.

9. Let A be the centre of the circle which touches internally another circle whose centre is B. Join AB, produce it to meet the larger circle at D, the smaller circle at C. Bisect CD at E. E is the centre of the required circle.

10. Draw the diagonals, and the point of intersection is the centre of the required circle.

11. Let ABC be the given sector, AB, BC being the given radii. Bisect the angle ABC by BD, and let BD meet the arc AC at D. Through D draw FDE at right angles to BD. Produce BA, BC to meet FDE at F, E respectively. Bisect the angle BED by EH meeting BD at the point H. H is the centre of the required circle.

12. With a radius equal to the sum of the radius of the given circle and that of the required one, describe a concentric circle cutting the given straight line, etc.

13. Towards the side in which the circle is situated, draw, at a distance of the given radius, a straight line parallel to the given straight line; then apply the preceding construction.

14. See Addl. Prop. VI. page 281.

15. *Let A be the given point, and C the given point in the* given straight line BC. From C draw CD at right angles to BC; at A in CA make the angle CAD equal to the angle DCA. D is the centre of the required circle.

16. Let AB, CD be the two given chords in the circles AB, CD. Find O, E the centres of the circles AB, CD. From O, E draw OF and EG perpendiculars to AB, CD respectively. With OF, EG as radii, describe two concentric circles. To these inner circles, draw a common tangent, and produce it to meet the original circles. [Addl. Prop. VI. page 281.]

17. From the given point A, remote from the given circle, draw AC perpendicular to the given straight line AB and make it equal to the radius of the given circle. Find D the centre of

the circle. Join CD. At D in CD make the angle CDE equal to the angle at C, and let CA produced meet DE at E. E is the centre of the required circle.

18. Let ABC be the triangle. Bisect the angles at A, B by AD and BD. From D draw DE, DF, DG, perpendiculars to AB, BC, CA, respectively. AG may be proved equal to AE, etc. E, F, G, are the points of contact.

19. Let AB be the given straight line. Take any point B, and draw BC at right angles to AB, making BC equal to the difference of the radii. From C as centre, and radius equal to the sum of the radii, describe a circle cutting BA in A. Join AC. Produce BC to E, making BE equal to the larger radius. Draw AD at right angles to AB, and make it equal to the smaller radius. D, E are the centres of the required circles.

20. Join P, Q with the centres and apply Euc. III. 18.

21. Bisect OP in A. Draw AB touching the circle PX, BA produced shall touch the circle SR.

22. Let A be the given centre, and B the centre of the given circle. Join AB. Draw a diameter at right angles to AB. The required circle passes through the extremities of this diameter.

23. At O the centre of the given circle draw two radii making an angle double the angle of an equilateral triangle, and meeting the circle at A, B. Draw tangents at A, B meeting each other at C. Join AB.

24. Apply Ex. 1, Prop. 17. 25. Apply Ex. 1, Prop. 17 and Euc. III. 31.

26. Let A, B be the centres of the circles, CD be the common tangent. Draw the radii AC, BD. From the centre of the smaller circle, draw a perpendicular to the radius of the larger circle. Apply Euc. II. 5, &c.

27. Let AB be the diameter. Produce AB to C so that the rect. AC, CB = the sq. on the given line. C is the point.

28. This is the same as Ex. 26. 29. Apply Euc. III. 14.

30. Let A, B, C be the three given points. Join AB, BC and bisect them in F, G respectively. At F, G draw FH, GK perpendiculars to AB, BC meeting the tangent through B at H, K. H, K are the centres of the required circles.

31. Draw any tangent AB touching the given circle at A. Make AB equal to the second straight line. From O as centre and with OB as radius, describe a circle cutting the first line at C. C is the required point.

32. Draw a common tangent through the point of contact of the two circles. Apply Ex. 1, Prop. 17.

33. Apply Euc. III. 20. 34. Apply Euc. III. 21.

35. This is the same as the first case of Ex 33.

36 Apply Euc. III 21 37 Apply Euc III 21, 26.

38 Apply Euc III. 21, 28 39 Apply Euc III. 22, 31

40 Apply Euc III 21 41 Apply Euc I 32 and III. 22.

42 The angle FEC is equal to the angle EAC, but EAC is equal to FFD, etc

43 A circle may be described about $BFED$ Apply Euc. III. 21, 32

44 & 45 Apply Euc III 22

46 Draw two radii making a right angle between them.

47 Apply Euc III 21 22, and I 26

48 Draw two radii making an angle equal to the supplement of the given angle Draw tangents from the extremities of the radii.

49 Let ABC be the larger circle draw two radii DB, DC at right angles to each other Join BC Draw DF perpendicular to BC From D as centre and with DF as radius describe a circle

50 Draw a common tangent through the point of contact and apply Euc III 32

51 Through P draw a common tangent TPT'

$$\angle DPC = \angle T'PC = \angle TPD = \angle PBC = \angle PAC = \angle APB$$

52 Let AE be the chord produced to C Let O be the centre, and let EO cut the circumference at D, and again at L when produced $\angle AEO = \angle EOC + \angle OBE$. $\angle AEL = 3\angle OBE = 3\angle OEB$

53 Apply Euc III 22 54 Apply Euc III 21 22

55 Apply Euc III 32

56 Apply Euc III 21 27 57 Apply Euc III 22

58 Describe a circle passing through the points A, D, C Produce DB to meet the circumference at the point E E is the other fixed point

59 Apply Euc III 26 and 27

60 Apply Euc III 28 and 29

61 A segment may be described on the same base such that all the vertices lie on the circumference of the segment Complete the circle (Euc III 25) The straight lines bisecting the vertical angles shall pass through the middle point of the arc, which subtends equal vertical angles

62 Produce PT to meet the circle ABP at M AMB or AQB is the supplement of the angle APB Hence BQR and AQS are perpendiculars on AP BP, respectively A circle can be described about $ARQT$ Therefore the angle RQA is equal to the angle RTA. Likewise SQB is equal to STB

63 Let A be the vertex of the given triangle ABC Produce BA to E, making AE equal to AC On AE describe a triangle ADE, so that AD may be equal to AB and ED to BC Let ED, BC, meet at F. A circle may be described about $ABFD$ (III 22)

64 Let A, B be the points of intersection of the circles, and let CBD cut them both In the arc opposite to ABD, take any point E Join EA, ED The angle AED is equal to the angle ACB, etc

65 Apply Euc III 28 and 27

66 On BC describe a segment of a circle BAC containing an angle equal to the vertical angle The vertices of all the triangles shall lie on the circumference of this segment Join BA, AC, BAC is one of the triangles On BC describe a segment BGC, containing an angle equal to the supplement of the angle BAC From B, C draw BE, CF perpendiculars on AC, AB respectively intersecting each other at the point D The point D lies on the circumference of the segment BGC For a circle may be described about $AFDE$, and the angle FDE or BDC is the supplement of A The line bisecting the vertical angle, shall pass through the middle point of the opposite arc of the circle BDC

67 Join the middle point of the circumference with the centre Apply Euc I 47

68 From D a point in the diameter AB produced, the secant DEF is drawn, so that DE = radius of the circle Let C be the centre. Join EC, FC [illegible] $CFD +$ [illegible] $= 3$ [illegible]

69 Let D be the centre of the circle and AB, AC the tangents A circle may be described about $BDCA$

70 This follows from the following proposition Of all triangles inscribed in a circle on a diameter as base, an isosceles triangle is the greatest

71 Let AB be the chord AC the tangent and D the middle point of the arc AB Join BD, AD Apply Euc III 32

72 Let A be the centre of the circle and B the given point Join AB and on it describe a semicircle ADB From the centre B and with radius equal to half the given difference describe a circle, cutting ADB at D Join BD and produce it to meet the original circle, cutting at E, F. EF is the required chord

73. Apply Euc III 31

74. AB, AC are drawn from the point of contact A, at right angles to each other meeting the circles at B, C Let DAE be the line joining the centres Join DB, EC The sum of the angles BAD, CAE is equal to a right angle.

75. Apply Euc III 31

76 Let C be the given point in the given straight line AB; A, B the other two points. From C draw CE at right angles to

AB. On *AB* describe a semicircle, cutting *CE* at *E*. From the centre *E*, and with radius *EC*, describe a circle. This circle is the required one.

77. Join the point in which the greater perpendicular cuts the circle, with the other extremity of the diameter.

Apply Euc. III. 31.

78. Apply Euc. III. 31.

79. Let *AB* be the given straight line, *EKI* be the given circle and *D* the given point within the circle. Find *C* the centre; join *CD*, and produce it to meet the circumference at *E*. At *E* and *C* make ½ right angles *CEI*, *ECF*. With *C* as the centre and *CF* as radius, describe the circle *HF*. Bisect *CD* at *G* and draw $GH \perp CD$, meeting the circle *FH* at *H*. With *G* as centre and *GH* as radius, describe a circle cutting the perpendicular from *C* on *AB* at *O*. Draw the chord *KOL* ∥ *AB*. *KL* is the required chord.

80. *SMR*, *SPR* are right angles (III. 31). Apply Euc. II. 12 and 3.

81. Apply Euc. III. 31.

82. Apply Ex. 73.

83. From *B* draw a straight line at right angles to *AC*, meeting the circle at *D*. On *DC* as diameter describe a circle.

84. See Ex. 5 of III. 31.

Let the chords *AB*, *CD* intersect each other at right angles at the point *E* (externally). Draw *CF* a diameter. Join *AF*, *AC*, *CB*, *BD*. The angle *FAC* is a right angle (III. 31), and the triangle *FAC* is equiangular to *CEB*. Hence *DB* is equal to *AF*.

85. Let *A* be the given point. Find the centre *B*. Join *AB*. On *AB* as diameter describe the circle *AEB*. Take any point *E* in the circle *AEB*. Join *AE* and produce it to meet the given circle at *C*. Join *EB*. *AEB* is a right angle (III. 31).

86. Let *AB* be the diameter, *BC* a tangent at *B*. At the point *A* in *BA*, make an angle equal to half a right angle, cutting the circle at *D* and the tangent at *C*.

87. Let *A* be the common centre. Take *AB* any radius of the inner circle. Produce *AB* to *C* making *BC* equal to *AB*. On *BC* as diameter describe a semicircle cutting the outer circle at *D*. Join *DB* and produce it to *F*, cutting the inner circle again at *E*, and meeting the other circle at *F*. *DF* is the required line.

88. Apply Euc. III. 31 and II. 12.

89. Apply Euc. III. 31.

90. A circle may be described about *AECF* (III. 31). Apply Euc. III. 22.

91 Let A, B be the given points in the given line AB. On AB describe the semicircle ACB cutting at C a line parallel to AB and at a distance equal to the given radius. Join AC, BC. From C draw CD perpendicular to AB. From the centre C and with radius CD describe a circle. From A, B draw tangents to the circle.

92 Apply Euc. III. 12 and 31.

93 Draw a tangent at the point of contact and apply Euc. III. 32.

94 Proceed as in Ex. 93.

95 Apply Euc. III. 32.

96 The tangents will form two parallelograms with the four sides of the quadrilateral produced (III. 32).

97 and 98 Apply Euc. III. 32.

99 A circle may be described about $ABDC$. Apply Euc. III. 35.

100 Apply Euc. III. 22, 25.

101 A circle may be described passing through the extremities of any of the tangents, the centre of the circle, and the point of intersection.

102 and 103 Apply Euc. III. 32.

104 Apply Euc. III. 31.

105 Let AB be the given base; on AB describe a segment of a circle, containing an angle equal to half the given vertical angle. From the centre A and with radius equal to the sum of the sides, describe a circle cutting the segment at D. Join AD. On AB describe a segment of a circle containing an angle equal to the vertical angle, and cutting AD at C. Join CB. ACB is the required triangle.

106 On the given base BC, describe a segment of a circle containing an angle equal to the vertical angle. At B draw BD at right angles to BC, and make it equal to the given altitude. Through D draw DA parallel to BC cutting the segment at A. ABC is the triangle required.

107 Let AB be the given base. On AB describe the segment of a circle ACB, containing an angle equal to the sum of a right angle and half the vertical angle. From the centre A, and with radius equal to the given difference, describe a circle cutting the segment ACB at C. Join AC, CB. Bisect BC at D. From D draw DE at right angles to CB, meeting AC produced at E. Join EB. AEB is the required triangle.

108 Let AB be the given base, and C the given point in it. On AB describe ADB a segment of a circle, containing an angle

equal to the given vertical angle. From C draw CE at right angles to AB, meeting the segment at E. AEB is the triangle required.

109. Find A, B the centres of the circles. From A any of these centres as centre, and with radius equal to the sum of the radii, describe a circle. From B draw BC touching this circle at C. Join AC cutting the circumference of the first circle at D. From B draw the radius BK, making it parallel to CA. Join KD. KD is the required tangent.

110. Let BAC be the vertical angle. Take AB, AC each equal to half the given perimeter. From B, C draw BD, CD at right angles to AB, AC respectively, and meeting each other at D. Join AD. BD may be proved equal to CD. From the centre D and with radius BD or DC describe the circle BHC. From the centre A and with radius equal to the given altitude describe the circle EGF. Draw a common tangent GH to these circles, so that the circles may remain on the opposite sides of GH (Ex. 109). Let GH, produced if necessary, cut AB at M and AC at N. AMN is the triangle required.

111. Describe the segment BAC on the base BC so that it may contain an angle = the given vertical angle. Bisect BC at D. With D as centre and radius = the given median, describe a circle cutting the segment BAC at A. BAC is the required triangle.

112. Let AB be the given base and C its middle point. On AB describe ADB a segment of a circle containing an angle equal to the vertical angle. From the centre C and with radius equal to the bisecting line describe a circle cutting the segment ADB at D. Join AD, BD.

113. Let AB be the given side. On AB describe an isosceles triangle ACB so that each of AC, BC will be equal to the given radius. From the centre C and with radius equal to CA or CB, describe the circle ABD. At A in BA make the angle BAD equal to the given angle, cutting the circle at D. Join BD. BAD is the triangle required.

114. Let A be the centre of the given circle. Draw two radii AB, AC making the angle BAC equal to two-thirds of a right angle. Join BC.

115. Let A be the centre of the given circle. Draw two radii AB, AC making the angle BAC equal to the supplement of the given angle. Join BC.

116. Let A, B, C be the given points. Join them so as to form a triangle. On AB, BC and towards the angles of the triangle opposite to them, describe two segments of circles containing angles, each equal to two-thirds of a right angle, and cutting each other at the point D. Join DA, DC, DB.

117. Apply Euc III. 32.

140 Let AB be the given chord, cut at any point C by the inner circle From the centre O, draw OD perpendicular to AB. The rectangle contained by AC, CB is equal to the difference of the squares on OB, OC.

141 Let AB and BC be the given straight lines, and D a given point in BC Draw DE at right angles to BC With the centre B and radius equal to the given straight line, describe a circle cutting DE at E Draw EC at right angles to EB, meeting BC at C* From F the middle point of CD, draw FG at right angles to BC, meeting AB at G G is the centre of the required circle

142 Let ABC be the given triangle inscribed in the circle ABC, AD perpendicular to the base BC, and AE a diameter Produce BA, EA to F, G, and make AF equal to AC, and AG to AD Join GF, BE, FC The angle BAF is equal to the angle BCE (III 21) The angle BCE is the complement of DCA (III 31), also DAC is the complement of ACD (I 32) Hence the angle GAF is equal to DAC Therefore AGF is a right angle (I 4) Also EBA is a right angle (III 31) Therefore a circle may be described about $EBGF$ Apply Euc III 35

143 Let AB, AC be the two given straight lines, and D the given point Bisect the angle BAC by AO Describe a circle which shall have its centre in AO, shall pass through D, and touch AB or AC (Ex 138)

144 Let AB be the given straight line, C the given point, and DEF the given circle Draw CG perpendicular to AB Produce CG to H, making GH equal to CG Describe a circle passing through H, C and cutting the given ⊙ at D and E Join DE and produce it to meet HC produced at K Draw KK touching the ⊙ DEF at K The circle described about HCK is the required ⊙ [III 36, Cor and 37]

145 Let AB be the diameter of the circle, from B the straight line BC is drawn touching the circle From the centre B and with radius equal to the side of the given square, describe a circle cutting the circle at D Join AD Produce AD to meet BC at C.

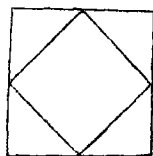
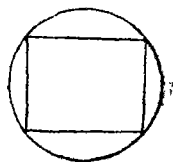
* If BE be less than BD, let the circle cut the semicircle on BD at E, and draw EC perpendicular to BD.

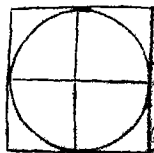
BOOK IV.

DEFINITIONS.

1. **Rectilineal figures** are those which are bounded by straight lines

2 A rectilineal figure is said to be **inscribed in** another rectilineal figure, when all the angles of the inscribed figure are on the sides of the figure in which it is inscribed, each on each

3 In like manner, one rectilineal figure is said to be **described about** another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each

4. A rectilineal figure is said to be **inscribed in** a circle when all the angular points of the inscribed figure are on the circumference of the circle

5 A circle is said to be **described about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described

6. A rectilineal figure is said to be **described about** a circle, when each side of the figure touches the circumference of the circle

7. In like manner, a circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle touches each side of the figure.

NOTE It might be remarked that the last six definitions state only three independant facts, but as each of these present two different points of view—*viz* that of the outer figure or that of the inner figure—we obtain six definitions altogether

Thus, Defs. 6 and 7 state the same fact, *viz* a circle within a rectilineal figure, but if we consider the circle to be given and wish to define the rectilineal figure described about it, then we have Def 6, on the other hand if we consider the rectilineal figure to be given and wish to define the circle drawn within it, then we get Def 7

This is what is meant by the *subjective* and *objective* view of the same fact Thus the subjective view of the rectilineal figure and the objective view of the circle gives us Def 6, but the subjective view of the circle and the objective view of the rectilineal figure, gives us Def 7

8 A straight line is said to be **placed in** a circle, when the extremities of it are on the circumference of the circle

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The Fourth Book of the Elements contains entirely a series of problems By some of these, circles may be inscribed and circumscribed in or about triangles, squares, and regular polygons, by others, triangles, squares, and some particular regular polygons may be inscribed and circumscribed in or about circles

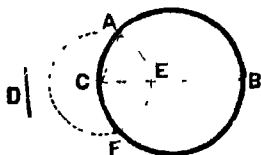
A rectilineal figure is called **equilateral**, when all its sides are equal to one another, and **equiangular**, when all its angles are equal to one another A **regular polygon** is that which is both equilateral and equiangular

OBS The student must not infer that because an equilateral triangle is equiangular, an equilateral polygon is likewise equiangular, for instance a rhombus is equilateral but not equiangular.

A polygon of five sides and angles is called a **pentagon**, of six sides and angles a **hexagon**, of seven sides and angles, a **heptagon**; of eight sides and angles, an **octagon**, of nine sides and angles, a **nonagon**, of ten sides and angles, a **decagon**, of eleven sides and angles, an **undecagon**, of twelve sides and angles, a **duodecagon**, and of fifteen sides and angles, a **quindecagon** or **pentadecagon**

Proposition 1. Problem

In a given circle, to place a chord, equal to a given straight line, which is not greater than the diameter of the circle

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle, *it is required to place in the circle* ABC, *a chord equal to* D.

Draw BC, a diameter of the ⊙ ABC.

Then, if BC = D, the thing required is done,

for in the ⊙ ABC, a chord BC is placed equal to D.

But, if it is not, BC is greater than D [*Hyp.*

Make CE = D, [I. 3.

and from the centre C, at the distance CE, describe the ⊙ AEF, cutting the given ⊙ at A, and join CA,

then CA shall be equal to D

∵ C is the centre of the ⊙ AEF,

∴ CA = CE, being radii,

= the given line D

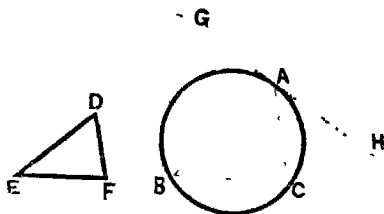
Wherefore, *in the circle* ABC, *a chord* CA *is placed equal to the given straight line* D, *which is not greater than the diameter of the circle* Q E F.

EXERCISE

To draw the diameter of a circle which shall pass at a given distance from a given point

Proposition 2 Problem.

In a given circle to inscribe a triangle equiangular to a given triangle

Let ABC be the given circle, and DEF the given triangle: *it is required to inscribe in the circle* ABC *a triangle equiangular to the triangle* DEF

At any point A on the ○ce draw a tangent GAH [III 17.

At A make the ∠ GAB = the ∠ F, [I 23.

also, make the ∠ HAC = the ∠ E

Join BC

Then ABC is the △ required

Since the chord AB is drawn from the point of contact A of the tangent GAH,

∴ the ∠ GAB = the ∠ C in the alternate segment [III 32.

But the ∠ GAB = the ∠ F, [*Cons.*

∴ the ∠ C = the ∠ F.

Similarly, the ∠ B = the ∠ HAC = the ∠ E.

∴ in the △s ABC, DEF,

the ∠ B = the ∠ E,

the ∠ C = the ∠ F,

∴ the third ∠ A = the third ∠ D [I. 32.

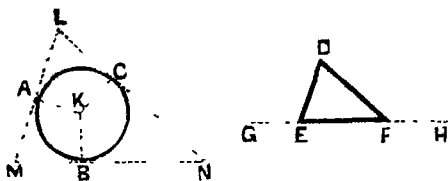
Wherefore, *the triangle* ABC *is equiangular to the triangle* DEF, *and it is inscribed in the circle* ABC Q E F.

EXERCISE

If an equilateral triangle be inscribed in a circle, show that the radii of the circle, drawn to the angular points, bisect the angles of the triangle.

Proposition 3. Problem.

About a given circle, to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: *it is required to describe a triangle about the circle* ABC *equiangular to the triangle* DEF

Produce EF both ways to the points G, H;

find the centre K of the ⊙ ABC, [III 1.

from K draw any radius KB

At K make the ∠ BKA = the ∠ DEG, [I. 23.

also, make the ∠ BKC = the ∠ DFH

At A, B, C draw the tangents LAM, MBN, NCL, respectively [III. 17,

Then LMN is the △ required.

Because in the quadrilateral AMBK,

the sum of the four angles = four right angles, [I 32 *Cor.*

and the sum of the ∠s at A and B = two right angles, [III 18.

∴ the sum of the ∠s at M and K = two right angles

But the sum of the ∠s DEF, DEG = two right angles,

and the ∠ DEG = the ∠ BKA,

∴ the ∠ M = the ∠ DEF

Similarly, the ∠ N = the ∠ DFE.

∴ in the △s LMN, DEF,

the third ∠ L = the third ∠ D. [I 32.

Wherefore, *the triangle* LMN *is equiangular to the triangle* DEF, *and it is described about the circle* ABC Q E F.

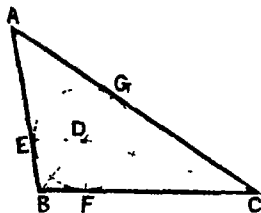
EXERCISES

1. An equilateral triangle inscribed in a circle is a fourth part of an equilateral triangle circumscribed about the same circle.

2. If a triangle be described about a given circle, the rectangle contained by the perimeter of the triangle and the radius of the circle is double the area of the triangle.

Proposition 4. Problem

To inscribe a circle in a given triangle.

Let ABC be the given triangle,
it is required to inscribe a circle in the triangle ABC.

Bisect the ∠s ABC, ACB, by the straight lines BD, CD, meeting one another at the point D, [I 9
and from D draw DE, DF, DG, perp to AB, BC, CA. [I 12.

Then, because in the △s DBE, DBF,
the ∠ DBE = the ∠ DBF, [*Constr.*
the ∠ DEB = the ∠ DFB, being right angles,
and DB is common
∴ DE = DF [I 26.
Similarly, DF = DG
∴ DE = DF = DG

Now, with centre D and radius DE, describe a ⊙.
This ⊙ must pass through the points E, F, G, and moreover must touch AB, BC, CA, at these points,
the angles at E, F, G, being right angles [III 16. *Cor.*
∴ the ⊙ is inscribed in the △ ABC

Wherefore, *a circle has been inscribed in the given triangle*
Q E F.

EXERCISES

1 DA bisects the angle BAC, hence show that the bisectors of the three angles of a triangle are concurrent
[Compare with Prop. XVI, page 101.

2 To describe a circle which shall touch the base of a given triangle and the other sides produced.
[Compare with Prop. XIX, page 104.

3. If a circle can be inscribed in a quadrilateral figure, the sums of its opposite sides are equal

4 If BC, CA and AB be represented by a, b, c, respectively, and half their sum by s, the distances of A, B, C from the points of contact are respectively $s-a$, $s-b$, $s-c$

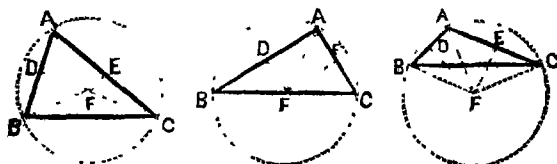
5 In fig Ex 2, the distances of A, B, C from the points of contact of the escribed circle that touches BC externally are s, $s-c$, $s-b$

6 Describe a circle touching two parallel straight lines and a third line which cuts the parallel lines

Proposition 5. Problem.

To describe a circle about a given triangle.

Let ABC be the given triangle

it is required to describe a circle about ABC

Bisect AB, AC at the points D, E, [I 10.

from these points draw DF, EF at right angles to AB, AC, [I 11.

DF, EF, produced will meet, being at right angles to AB, AC, which themselves meet

Let them meet at F

Join AF, also join BF and CF, if F be not in BC

Then, in the △s ADF, BDF,

AD = BD,

DF is common,

and the ∠ ADF = the ∠ BDF, being right angles:

∴ AF = BF,

Similarly, we may shew that AF = CF.

∴ AF = BF = CF

With centre F and radius FA describe a ⊙,

then this ⊙ must pass through A, B, C,

∴ it will be described about the △ ABC.

Wherefore, ***a circle has been described about the given triangle.***

Q. E. F.

COR. It is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle,

And when the centre is on one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle;

And if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle [III 31.

Therefore, conversely, if the given triangle be acute-angled, the centre of the circle falls within it, if it be a right-angled triangle, the centre is on the hypotenuse, and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle

EXERCISES

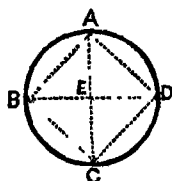
1. The perpendicular from F bisects BC, hence show that the three straight lines at right angles to the sides of a triangle at their middle points are concurrent (See Prop XV, page 100).

2 The diameter of the circle described about an equilateral triangle is double the diameter of the circle inscribed in the same triangle

Proposition 6. Problem.

To inscribe a square in a given circle.

Let ABCD be the given circle.

It is required to inscribe a square in ABCD

Draw two diameters AC, BD at right angles, intersecting at the centre E, [III 1, I 11.

and join AB, BC, CD, DA

The figure ABCD shall be the square required.

In the $\triangle$s AEB, AED,

EB=ED, being radii of the $\odot$,

EA is common,

and the $\angle$ AEB = the $\angle$ AED, being right angles:

$\therefore$ BA = DA [I. 4.

Similarly, we may shew that

BA = BC = CD = DA.

$\therefore$ the figure ABCD is equilateral.

And since the diameter AC divides the $\odot$ into two semicircles,

$\therefore$ each of the $\angle$s ABC, ADC, is a right angle. [III. 31.

Similarly, each of the $\angle$s BAD, BCD, is a right angle.

$\therefore$ the figure ABCD is a square,

and it has been inscribed in the given $\odot$.

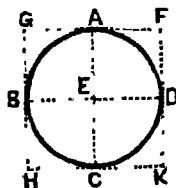
Wherefore, ***a square has been inscribed in the given circle.*** **Q.E.F.**

EXERCISE

The inscribed square is equal to twice the square on the radius or to half the square on the diameter.

Proposition 7. Problem.

To describe a square about a given circle.

Let ABCD be the given circle.
it is required to describe a square about it.

Draw two diameters, AC, BD at right angles, intersecting at the centre E, [III 1, I 11.
and through A, B, C, D, draw FG, GH, HK, KF, touching the ⊙ [III. 17

The figure GHKF shall be the square required

Since the radius EA meets the tangent GAF at A the point of contact,

∴ the ∠s at A are right angles [III 18

Similarly, the ∠s at B, C, D, are also right angles

And since the ∠s AEB, EBG, are both right angles,

∴ AC is par[l] to GH [I 29.

Likewise AC is par[l] to FK

Similarly, it may be shewn that BD is par[l] to GF and HK

∴ each of the figs AH, AK, BF, BK, and also GK are par[ms].

∴ **GH** = **FK** = AC = BD = **GF** = **HK**.

∴ GHKF is equilateral

Again, since AH is a par[m],

∴ the ∠ H = the ∠ CAG [I. 34.

= a right angle.

Similarly, the ∠s at G, F, K, are also right angles

∴ GHKF is a square

Wherefore, *a square has been described about the given circle.*

Q E F.

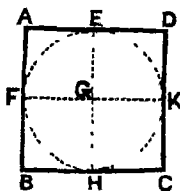
EXERCISE

The circumscribed square is double of the inscribed square or four times the square on the radius.

Proposition 8. Problem.

To inscribe a circle in a given square.

Let ABCD be the given square

it is required to inscribe a circle in ABCD

Bisect the sides AB, AD, at the points F, E, [I 10

through E draw EH par[l] to AB or DC, [I 31

and through F draw FK par[l] to AD or BC,

EH cutting FK at G

Then each of the figs GA, GB, GC, GD, is a rectangle.

Since AB = AD, being the sides of a square,

their halves are also equal,

∴ AF = AE,

∴ GE = GF, since GA is a rectangle

Similarly, we may shew that

GH = GK = GE = GF.

Now, with centre G and radius GE, describe a ⊙,

this ⊙ must pass through the points E, F, H, K,

and touch the sides of the square at these points,

[III 16 *Cor.*

since the ∠s at these points are right angles [I 29

∴ the ⊙ EFHK has been inscribed in the square ABCD.

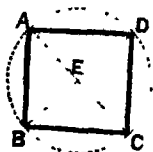
Wherefore, *a circle has been inscribed in the given square*

Q. E. F.

Proposition 9. Problem.

To describe a circle about a given square.

Let ABCD be the given square

it is required to describe a circle about ABCD.

Draw the diagonals AC, BD, intersecting at E.

Then in the △s ABC, ADC,

AB = AD,

AC is common,

and BC = DC

∴ the ∠ BAC = the ∠ DAC,

∴ the diagonal AC bisects the ∠ BAD

In like manner it may be shewn that the other ∠s of the square, are bisected by the diagonals AC, BD,

that is, each of the two ∠s at A, B, C, D, is half a right angle.

∴ EA = EB, [I 6.

EB = EC,

EC = ED,

ED = EA

∴ EA = EB = EC = ED

Now, with centre E and radius EA, describe a ⊙,

this ⊙ will pass through the points A, B, C, D,

and ∴ will be described about the square ABCD.

Wherefore, *a circle has been described about a given square.*

Q. E. F.

Alternative Solution.

On any diagonal BD as diameter, describe two semicircles.

These semicircles must pass through A, C, respectively,

since the ∠s BAD, BCD, are right angles [III 31.

But since these two semicircles have a common diameter BD,

they must together form one complete circle.

∴ the ⊙ described on any diagonal BD as diameter,

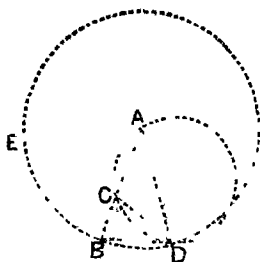
passes through the four points A, B, C, D, of the given square.

EXERCISE.

To describe a circle about a given rectangle.

Proposition 10 Problem.

To describe an isosceles triangle, having each of the angles at the base double of the third angle

Take any straight line AB, and divide it at C
so that the rect AB, BC = the sq on AC. [II. 11.

With centre A, and radius AB, describe a ⊙ ABE,
in which place the chord BD equal to AC [IV. 1.
Join AD

Then ABD *is the* △ *required*

Join CD, and about the △ ACD describe a ⊙. [IV. 5.
Then, the rect AB, BC = the sq on AC, [*Constr.*
= the sq on BD [*Constr.*
∴ BD is a tangent to the ⊙ ACD [III. 37.
And since DC is a chord through its point of contact,
∴ the ∠ CDB = the ∠ A, in the alternate segment. [III. 32.
To each add the ∠ ADC
∴ the whole ∠ ADB = the sum of the ∠s A and ADC,
= the exterior ∠ BCD. [I. 32.
But the ∠ ADB = the ∠ ABD, [I. 5.
since AB = AD, being radii :
∴ the ∠ ABD = the ∠ BCD,
∴ CD = BD [I. 6.
= CA : [*Constr.*
∴ the ∠ A = the ∠ ADC ; [I. 5.

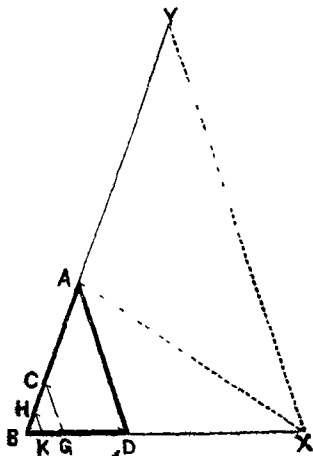
∴ the sum of the ∠s A and ADC = double the ∠ A.
But, the sum of the ∠s A and ADC = the ∠ ADB: [*Proved.*

∴ the ∠ ADB, or the ∠ ABD = double the ∠ A.

Wherefore, *an isosceles triangle* ABD *has been described, having each of the angles at the base double of the third angle.*

Q. E. F.

OBS. The Δ BCD is also such an isosceles triangle. Hence we may infer the following most curious result:—

Take an isosceles Δ ABD (such as in this Prop.),

and from the side AB cut off AC = BD, thus forming the Δ BCD; then from the side DB cut off DG = BC, thus forming the Δ BGC; again from the side BC cut of CH = BG, thus forming the Δ BHG; and similarly form the Δ BKH, and so on. Hence we see that from the given isosceles Δ ABD, we may cut off successively a series of diminishing isosceles Δs BCD, BGC, BHG, BKH, etc., all similar* to the original Δ, and if we continue this operation long enough, the series of Δs will dwindle down to a point at B (if we are careful in making the section only on those sides of the series

* Two triangles are similar when they are equiangular. Thus in Prop. 2, ABC and DEF are similar triangles.

which are adjacent to B, and by a like treatment we may cause the series to vanish at any of the points D, C, G, H, K, etc., we please)

In a like manner, if in the original △ we produce BA, BD, and take DX=BA, then the △ BAX is similar to the △ ABD; again if AY=BX, then the △ BXY is also similar to the △ ABD, and so on we obtain a series of larger and larger isosceles △s, similar to the original △ ABD, and all diverging away from the angular point B of the original △ ABD

We may observe that

. KH∥GC∥DA∥XY .etc.
GH∥DC∥XA . etc

Hence, once we have found the position of C in the given △ ABD, the rest of the construction in these series consists of drawing parallels to CD and AD, respectively

Finally,

BC is divided "in extreme and mean ratio" at H,
BA so divided at C, BY at A, etc., also
BG so divided at K, BD at G, BX at D, etc.

Hence a straight line BH may be produced successively to the points C, A, Y, etc., so that

$$BH.BC=HC^2,\ BC.BA=CA^2,\ BA.BY=AY^2,\ \text{etc}$$

EXERCISES

1 Show that BD is equal to the side of a regular pentagon inscribed in the smaller circle, and a side of a regular decagon inscribed in the larger circle

2. To describe a regular decagon on a given finite straight line.

3 If the smaller circle cut the larger in a second point F, show that the triangle ADF is congruent with the triangle ABD

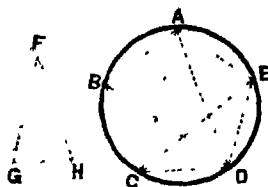
4 Divide a right angle into five equal parts

5 Divide an isosceles triangle (such as in this Prop.) by a straight line through one of its angular points into two other isosceles triangles, but one of which will be similar to the original triangle, and the other will have the vertical angle three times each of the base-angles

Proposition 11 Problem.

To inscribe a regular pentagon in a given circle.

Let ABCDE be the given circle.

It is required to inscribe a regular pentagon in the circle ABCDE.

Describe an isosceles △ FGH, having each of the angles at G, H, double of the angle at F [IV. 10

In the ⊙ ABCDE, inscribe the △ ACD, equiangular to the △ FGH, [IV. 2.

so that each of the ∠s ACD, ADC, is double of the ∠ CAD.

Bisect the ∠s ACD, ADC, by CE and DB, [I. 9.

meeting the ○ee at E and B

Join EA, ED, BA, BC

Then ABCDE is the regular pentagon required.

Since the ∠s ACD, ADC, are each double of the ∠ CAD,

and they are bisected by CE, DB :

∴ the five ∠s CAD, DCE, ECA, ADB, BDC, are all equal.

∴ the five arcs CD, DE, EA, AB, BC, are all equal.

[III. 26.

∴ the five chords CD, DE, EA, AB, BC, are all equal.

[III. 29.

∴ the pentagon is equilateral.

Again, since the five arcs are all equal,

∴ the sum of any three of them = the sum of any other three,

∴ the whole arc BCDE = the whole arc CDEA.

∴ the ∠s standing on these arcs are equal, [III. 27.

∴ the ∠ BAE = the ∠ CBA.

Similarly, the other angles of the pentagon may be proved to be equal

∴ the pentagon is equiangular.

Hence the pentagon is regular, being both equilateral and equiangular, and it is inscribed in the ⊙ ABCDE

Wherefore, *a regular pentagon has been inscribed in the given circle* Q E F

EXERCISES

1. To describe a regular pentagon on a given finite straight line

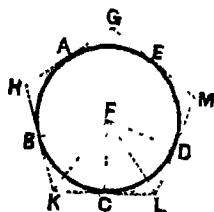
2 Each diagonal of a regular pentagon is parallel to the side with which it is not conterminous

3 If BE cuts AD at K, show that BCDK is a lozenge

4. The figure formed by the diagonals is a regular pentagon.

Proposition 12 Problem

To describe a regular pentagon about a given circle.

Let ABCDE be the given circle
it is required to describe a regular pentagon about the circle ABCDE

Let the angles of a pentagon, inscribed in the circle, by the last Proposition, be at the points A B, C, D, E, so that the arcs AB, BC, CD, DE, EA, are equal,

and through the points A, B, C, D E draw GH, HK, KL, LM, MG, touching the circle [III 17.

Then the figure GHKLM shall be the pentagon required.

Take the centre F, and join FB, FK, FC, FL, FD.

Then in the △s KFB, KFC

FB=FC, being radii of the ⊙,

KF is common,

and KB=KC, being tangents to the ⊙ from the same point K. [III 17.

∴ the two △s are identically equal, [I. 8.

∴ the ∠ KFB=the ∠ KFC,

that is, the ∠ BFC=double the ∠ KFC;

also, the ∠ BKF=the ∠ CKF,

that is, the ∠ BKC=double the ∠ CKF.

Likewise, it may be proved that

the ∠ CFD=double the ∠ CFL,

and the ∠ CLD=double the ∠ CLF.

But the ∠ BFC = the ∠ CFD,

since they stand on the equal arcs BC, CD, [III. 27.

∴ their halves are also equal,

∴ the ∠ KFC = the ∠ CFL

Hence, in the △s KFC, CFL,

the side CF is common,

the ∠ KFC = the ∠ CFL,

and the ∠ FCK = the ∠ FCL, being right angles; [III 18.

∴ KC = CL, [I 26.

and the ∠ CKF = the ∠ CLF

Hence we have KL double of KC,

and similarly we may shew that KH is double of KB,

∴ KL = KH, since KC = KB.

Also, we have shewn that the ∠ CKF = the ∠ CLF,

∴ their doubles are also equal,

∴ the ∠ BKC = the ∠ CLD

Similarly it may be shewn that

KL = LM = MG = GH = HK,

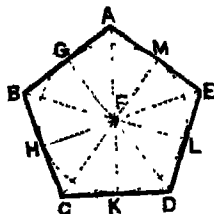
and that the ∠ CLD = ∠ M = ∠ G = ∠ H = ∠ BKC

∴ the pentagon GHKLM is both equilateral and equiangular, that is, it is regular, and has been described about the given circle

Wherefore, *an equilateral and equiangular pentagon has been described about a given circle* Q. E. F.

Proposition 13 Problem.

To inscribe a circle in a given regular pentagon.

Let ABCDE be the given regular pentagon;
it is required to inscribe a circle in the pentagon ABCDE.

Let CF and DF be the bisectors of the ∠s BCD and CDE, which are adjacent to any side CD of the pentagon, and let CF, DF intersect at F.
Join FB, FA, FE,
and from F drop FG, FH, FK, FL, FM, perp to the sides. [I. 12.

Then, in the △s FCB, FCD,
CB = CD,
CF is common,
and the ∠ FCB = the ∠ FCD [Constr.
∴ the ∠ CBF = the ∠ CDF, [I. 4.
= half the ∠ CDE, [Constr.
= half the ∠ CBA
∴ FB bisects the ∠ CBA.

Similarly, it may be shewn that FA and FE bisect the ∠s at A and E.

Again, in the △s FCH, FCK,
the ∠ FCH = the ∠ FCK,
the ∠ CHF = the ∠ CKF, being right angles,
and CF is common:
∴ FH = FK [I. 26.

In like manner it may be proved that the other perps. FL, FM, FG, are each equal to FH or FK.

Now, with centre F and radius FH, describe a ⊙.

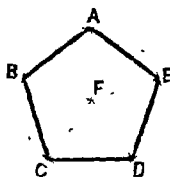
This ⊙ must pass through the points G, H, K, L, M, and touch the sides of the pentagon at these points, since the ∠s at these points are right angles.

Wherefore, *a circle has been inscribed in the given regular pentagon* Q. E. F.

Proposition 14 Problem

To describe a circle about a given regular pentagon.

Let ABCDE be the given regular pentagon

it is required to describe a circle about it

Bisect the ∠s BCD and CDE by CF and DF, which intersect at F Join FB, FA, FE

Then it may be shewn, as in the preceding Proposition, that FB, FA, FE, bisect the ∠s CBA, BAE, AED, respectively

∴ FA, FB, FC, FD, FE, bisect the five ∠s of the pentagon ,

∴ these lines divide the pentagon into five △s,

each of whose base-angles = half the ∠ of the regular pentagon ,

∴ these base-angles are all equal ;

∴ each of these five △s is isosceles , [I 6.

∴ FA = FB

FB = FC

FC = FD

FD = FE

FE = FA.

∴ FA = FB = FC = FD = FE = FA.

Now, describe a ⊙ with F as centre and radius equal to any of these lines, and this ⊙ will be described about the pentagon ABCDE

Wherefore, *a circle has been described about the given regular pentagon.* Q. E. F.

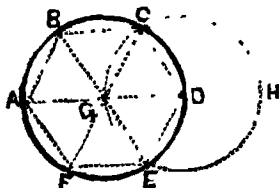
EXERCISE.

Shew that in every regular polygon, the bisectors of the angles and the perpendiculars through the middle points of the sides, are concurrent at the point which is the common centre of the inscribed and circumscribed circles

Proposition 15. Problem.

To inscribe a regular hexagon in a given circle.

Let ABCDEF be the given circle:

it is required to inscribe a regular hexagon in it.

Find the centre G of the ⊙ ABCDEF, [III 1.
and draw any diameter AGD.

With centre D and radius DG, describe a ⊙ EGCH,
cutting the other ⊙ at C and E

Join CG, EG, and produce them to meet the ○ce at F, B

Join AB, BC, CD, DE, EF, FA

Then ABCDEF shall be the hexagon required.

Then it may be proved, as in I 1, that
DGC, DGE, are equilateral △s.

∴ each of the ∠s DGC, DGE = one-third of two rt. angles. [I. 32.

But the sum of the ∠s DGC, DGE, EGF = two rt. angles; [I. 13.

∴ the remaining ∠ EGF = one-third of two rt. angles.

∴ the ∠s DGC, DGE, EGF, are all equal.

∴ their vertical ∠s AGF, AGB, BGC, are also equal to these. [I. 15.

∴ the six ∠s at the centre G are all equal;

∴ the six arcs AB, BC, CD, DE, EF, FA, are all equal; [III. 26.

∴ the six chords AB, BC, CD, DE, EF, FA, are all equal; [III. 29.

∴ the hexagon is equilateral.

Again, since the six arcs are all equal,

∴ any four of them together = any other four;

∴ the whole arc ABCDE = the whole arc BCDEF;

∴ the ∠s at the ○ᶜᵉ, which stand on these arcs, are also equal, that is, the ∠ EFA = the ∠ FAB. [III. 27.

Similarly, the other ∠s of the hexagon may be shewn to be equal

∴ the hexagon is also equiangular.

∴ the hexagon is regular, and has been inscribed in the given ⊙.

Wherefore, *a regular hexagon has been inscribed in the given circle.* Q. E. F.

COR From this it is manifest that the side of the hexagon is equal to the radius of the circle

Also, if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle a regular hexagon will be described about the circle, as may be shewn from what was said of the pentagon; and a circle may be inscribed in a given regular hexagon, and circumscribed about it, by a method like that used for the pentagon

EXERCISES

1 The straight lines joining AE, EC, CA form the inscribed equilateral triangle

2 To describe a regular hexagon on a given finite straight line

3. To describe a regular hexagon about a given circle.

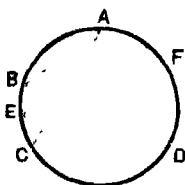
4 The side of a regular hexagon inscribed in a circle is equal to the radius, and the area of the hexagon is *six times* the area of an equilateral triangle constructed on the radius.

5. If the circumference of a circle be divided into six equal parts, at points A, B, C, D, E, F, successively, then the straight line joining BF, is trisected by the straight lines joining AC and AE.

Proposition 16 Problem

To inscribe a regular quindecagon in a given circle.

Let ABCD be the given circle

it is required to inscribe a regular quindecagon in the circle ABCD

Let AC be the side of an equilateral △ inscribed in the ⊙ [IV 2

and let AB be the side of a regular pentagon inscribed in the same ⊙ [IV 11

Then, since the whole ○ᶜᵉ will be divided into fifteen equal parts by the sides of the regular quindecagon

∴ the arc AC, which is a third of the whole ○ᶜᵉ, must contain five of these equal parts,

also, the arc AB, which is a fifth of the whole ○ᶜᵉ, must contain three of these equal parts,

∴ the difference of the arcs AC, AB, contains two such equal parts,

that is, the arc BC contains two such equal parts

Bisect the arc BC at E ; [III. 30.

∴ each of the arcs BE, EC, is a fifteenth part of the whole ○ᶜᵉ

Hence, if BE, EC, be joined, and chords equal to them be placed successively along the ○ᶜᵉ, a regular quindecagon will be inscribed in the ⊙. Q E F.

NOTES ON BOOK IV.

Prop 3 That the tangents at A, B and C, must meet and form a triangle may be easily proved by joining AB, BC, and CA. The angle MAB is less than the angle MAK (I. ax. 9), but MAK is a right angle (III 18), therefore the angle MAB is less than a right angle Likewise ABM is less than a right angle Therefore the two angles MAB and ABM are together less than two right angles, consequently the tangents AM and BM must meet one another Similarly it may be proved that AL and CL, as also CN, BN, must meet one another

The construction of this Proposition may be effected without producing EF Produce BK to G, and at the point K in the straight line GK and on both sides of it, make the angles GKA and GKC equal to the angles DEF and DFE, respectively

Prop 4 It is assumed in the construction of this Proposition that the bisectors of the angles B and C will meet This however may be proved by showing that they make angles with BC which are less than two right angles

By a process similar to that in Prop 4 we may describe three circles each touching one side externally and the other two sides produced the only difference in these cases would be to bisect the two exterior angles instead of the interior ones, through which the sides are produced

To express the radius of the circle inscribed in a triangle in terms of the sides of the triangle Join AD Let the sides opposite to the angles A, B and C be represented by a, b, and c, respectively, and let r represent the radius, and also let $s = \frac{1}{2}(a+b+c)$

The area of the triangle in terms of the sides

$= \sqrt{\{s(s-a)(s-b)(s-c)\}}$ [*See* Notes on Book II 13, p 188]

But again, the area of the triangle ABC

= areas of ADB, BDC, ADC

$= \frac{1}{2}AB \times r + \frac{1}{2}BC \times r + \frac{1}{2}AC \times r = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

$= \frac{1}{2}(a+b+c)r = sr$

$$\therefore \; sr = \sqrt{\{s(s-a)(s-b)(s-c)\}};$$

$$\therefore \; r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

in other words

$$r = \frac{\text{area of } \Delta}{\text{semi-perimeter}},$$

$$\therefore \text{ Area of } \Delta = sr$$

Example. Find the radius of a circle inscribed in the triangle whose sides are 13, 14, and 15 inches

$$s=\frac{1}{2}(13+14+15)=21;$$

$$\therefore\ r=\sqrt{\frac{8\times7\times6}{21}}=4 \text{ inches}$$

PROP 5 is the same as "To describe a circle that shall pass through three given points not in the same straight line"

The radius of the circumscribed circle may be expressed in terms of the sides of the triangle From A draw AP perpendicular to BC. Let the sides opposite to the angles A, B, C, be represented by a, b and c, respectively, and let R represent the radius, also let $s=\frac{1}{2}(a+b+c)$.

The area of the triangle ABC
$=\sqrt{\{s(s-a)(s-b)(s-c)\}}=sr$ [*See* Notes to Prop 4

But the area also $=\frac{1}{2}\times AP\times BC=\frac{1}{2}a\times AP$ [Euc. I. 41].

$$\therefore\ \tfrac{1}{2}a\times AP=Area\ ,\ \therefore\ AP=\frac{2\times Area}{a}$$

Again AB × AC = AP × diameter of the circumscribed circle,
[*See* Ex 142, p 300].

$$=AP\times 2R$$

$$\therefore\ bc=AP\times 2R.$$

$$\therefore\ 2R=\frac{bc}{AP}=bc-\frac{2Area}{a}=\frac{abc}{2\times Area}$$

$$\therefore\ R=\frac{abc}{4\times Area}=\frac{abc}{4sr}=\frac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}}.$$

Example. The sides of a triangle are 10, 21 and 17 inches; find the radius of the circle circumscribing the triangle

$$s=\frac{1}{2}(10+21+17)=24.$$

$$\therefore\ R=\frac{10\times21\times17}{4\sqrt{(24\times14\times3\times7)}}=\frac{10\times21\times17}{4\times7\times4\times3}=\frac{85}{8}=10\tfrac{5}{8}$$

PROP. 7. We may deduce the following :—The square described about a circle is four times the square on the radius of the circle

The figure formed by joining AB, BC, CD, DA is a square [IV 6. Hence we may deduce the following corollary :—The square described about a circle is double of the square inscribed in the same circle.

It is obvious that a rectangle as well as a square, may be inscribed in a circle, but no rectangle besides a square can be described about a circle

PROP. 10. The vertical angle of the triangle described, is the fifth part of two right angles. By bisecting this angle, we obtain the fifth part of a right angle. Hence a right angle may be divided into five equal parts.

PROP. 12. This Proposition is a particular case of the following:—*If tangents be drawn at the angular points of a regular polygon, inscribed in a circle, they will form a regular polygon of the same number of sides, circumscribing the circle* The demonstration is the same as in this Proposition.

PROP 13. As in this Proposition, a circle may be inscribed in any regular polygon

PROP. 14. As in this Proposition, a circle may be described about any regular polygon.

PROP 16. The construction of this Proposition may also be effected by placing in the circle from the same point of the circumference, two straight lines which are respectively equal to the sides of a regular hexagon and a regular pentagon, which may be inscribed in the circle, and then taking twice the difference of their arcs

The centre of the circle inscribed in a regular polygon is the same as that of the circle circumscribed about it

Every equilateral polygon inscribed in a circle is also equiangular. This is evident, because equal chords cut off equal arcs, and equal angles stand upon equal arcs, etc

Every equilateral polygon circumscribed about a circle is also equiangular This is best proved by symmetry, since every diameter passing through an angular point of the polygon is an axis of symmetry for the polygon

By inscribing an equilateral triangle, a square, a regular pentagon, a regular hexagon, and a regular quindecagon in a circle, the circumference is divided into three, four, five, six, and fifteen equal parts respectively By bisecting the arcs so obtained, we get a sixth, eighth or tenth, etc , part of the circumference of a circle , again bisecting them, twelfth, sixteenth, or twentieth, etc , part of the circumference But a geometrical method of dividing the circumference of a circle equally into any given number of parts, has never yet been discovered.

A celebrated theorem of Proclus concerning ordinate polygons should not be omited ; the theorem is, that a multiple of the angles of three regular figures only (namely, an equilateral triangle, a square, and a hexagon) can be so placed at a point as to fill up the space around it. For this purpose it is necessary that the angle of the polygon should be an aliquot part of four right angles, since the angles at any point are equal to four right angles , therefore, as the angle of an equilateral triangle is the sixth part of four right angles, the angle of a square the fourth part, and the angle of a hexagon the third part, it is evident, that six equilateral triangles, four squares, or three hexagons, can be so placed at a point as to make a continuous surface.

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QUESTIONS ON BOOK IV.

1 Of what sort of propositions does the Fourth Book of the Elements consist?

2 When is one rectilineal figure said to be inscribed in another rectilineal figure?

3 When is a rectilineal figure said to be inscribed in a circle, and when described about a circle?

4 When is a circle said to be inscribed in a rectilineal figure?

5. Define a regular polygon

6 Mention the regular figures which are not polygons

7 When is a straight line said to be placed in a circle?

8 In PROP. 3, show that the tangents at A, B, C meet one another and form a triangle

9 What is an escribed circle of a triangle? Give the construction of the problem by which it may be found

10 The sides of a triangle are 17, 25, 28 inches, find the radii of the inscribed and circumscribed circles

11 The sides of a triangle are given find in terms of the sides, the radius of the circle which touches one side and the other two sides produced

12 The sides of a triangle are ten, twenty-one, and seventeen inches find the radius of the escribed circle which touches the longest side externally and the other two sides produced

13 When does the centre of the circle inscribed in a circle coincide with that of the circle circumscribed about it?

14 Show that the radius of a circle inscribed in an equilateral triangle, is half of the radius of the circle described about the same triangle.

15 Show that the square described about a circle, is double of the inscribed square and four times the square of the radius.

16 What parallelograms can be inscribed in a circle, and in what parallelograms can circles be inscribed?

17 Show that in the Fig Prop 10, if the points of intersection of the circles be joined with the centre of the larger circle and with each other, another triangle will be formed equiangular and equal to the former

18 Divide a right angle into five equal parts.

19 Find the value of each of the interior angles of a regular pentagon.

20. Show that the centre of the circle inscribed in a regular pentagon is the same as that of the circle circumscribed about it.

21 The figure formed by joining the first, third, and fifth angles of a regular hexagon inscribed in a circle, will form an equilateral triangle which will also be inscribed in the same circle.

22 A side of an equilateral triangle inscribed in a circle is three units of length find the side of a regular hexagon inscribed in the same circle

23 By what propositions of *Euclid* can we divide the circumference of a circle into 10 and 20 equal parts?

24 By what propositions of *Euclid* can we divide the circumference of a circle into 4, 8, 16, etc, equal parts?

25 Find the value of each of the interior angles of a regular hexagon

26 A side of an equilateral triangle inscribed in a circle is 9 inches find the radii of the inscribed and circumscribed circles

27 The sides of a triangle are 25, 36, and 29 inches find the radii of the inscribed and circumscribed circles

28 The radius of a circle is eight inches find the area of a regular hexagon inscribed in it

29 The area of a regular hexagon inscribed in a circle is 45 square inches find the area of a regular hexagon circumscribed about it

30 Inscribe a regular octagon in a circle

31 Find the value of each of the angles of a regular octagon inscribed in a circle

32 A polygon is inscribed in a circle show that it is equiangular if it is equilateral, and also show that it is not always équilateral if it is equiangular

33 The alternate sides of a regular octagon are produced to meet what figure is thus formed?

34 A side of a regular octagon inscribed in a circle is four inches find the radius of the circle

35 The area of a square inscribed in a circle is 20 square inches find the area of a regular octagon inscribed in the same circle

36. Divide the circumference of a circle into ten equal parts

37 Find the value of each of the angles of a regular decagon inscribed in a circle

38 Find the area of a regular decagon, one of whose sides is 4 inches in length

39 Divide the circumference of a circle into twelve equal parts.

40. Find the value of each of the angles of a regular duodecagon inscribed in a circle.

41. A regular polygon inscribed in a circle has n sides: find the value of each of the angles of the figure.

42. Show that in a regular polygon of n sides, each angle is equal to $\frac{2}{n}(n-2)$ right angles.

43. What regular polygon has each of its angles equal to nine-tenths of two right angles?

44. Find the value of the angle subtended at the centre of a circle by an arc which is a twelfth part of the circumference.

45. Mention the three regular figures whose angular points can be so placed as to fill up the space around a point.

46. Find, in some other way besides that which is given by *Euclid*, the arc subtending the side of a regular quindecagon, inscribed in a circle

47. In Fig. Euc IV 10, how many degrees are there in the angle BAD, and of what regular figure that may be inscribed in the circle BED is the base BD a side?

48. Show how to *inscribe* a circle in a regular polygon of n sides.

49. Show how to describe a circle about a regular polygon of n sides.

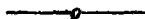
50. Show how to inscribe in, or to circumscribe about a given circle, regular figures of 8, 16, 32, .. sides

51 Show how to inscribe in, or to circumscribe about a given circle, regular figures of 12, 24, 48,. sides

52. Show how to inscribe in, or to circumscribe about a given circle, regular figures of 20, 40, 80, ...sides

53. Show how to inscribe in, or to circumscribe about a given circle, regular figures of 30, 60, 120..... .sides.

54 Find the values of the equal angles in the polygons mentioned in questions 50 and 51

55. Find the values of the equal angles in the regular figures mentioned in questions 52 and 53.

ADDITIONAL PROPOSITIONS. BOOK IV.

Proposition I. Theorem

If several triangles on the same base and on the same side of the base have equal vertical angles, the extremities of the base and the several vertices are concyclic

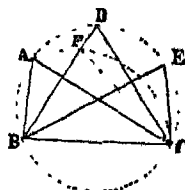
Let ABC, DBC, EBC, be △s on the same base BC, and on the same side of it,

the points B, C, A, D, E, *are concyclic*

Describe the ⊙ ABC about the △ ABC. [IV 5.

This ⊙ will pass through D If not, let the ⊙ ABC cut BD at F. Join FC

The ∠ BAC=the ∠ BFC [III 21.

Also, the ∠ BAC=the ∠ BDC [*Hyp*

∴ the ∠ BFC=the ∠ BDC [*Ax.* 1.

But the ∠ BFC is greater than the ∠ BDC; [I. 16.

which is impossible

∴ the ⊙ will not cut BD.

Similarly, the ⊙ will not cut BD produced.

∴ the ⊙ will pass through D

Likewise, we can prove that the ⊙ will pass through E, and through the vertices of all other △s on the same base BC, and on the same side of it, and having equal vertical angles

Wherefore, *if several triangles on the same base* &c. Q. E D.

Proposition II Theorem.

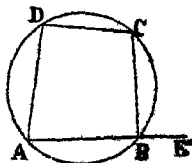
If a side of a cyclic quadrilateral be produced, the exterior angle is equal to the interior opposite angle.

Let ABCD be a quadrilateral inscribed in the ⊙ ABCD, AB is produced to E

The angle CBE *is equal to the angle* ADC.

The ∠ ABC is supplementary to the ∠ CBE; [I. 13.

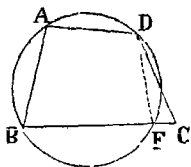
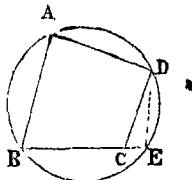
also, it is supplementary to ADC. [III. 22

∴ the ∠ CBE=the ∠ ADC.

Wherefore, *if a side of a cyclic quadrilateral* &c. Q. E. D.

Proposition III. Theorem

If any two opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic

Let ABCD be a quadrilateral, in which the ∠s BAD, BCD, are supplementary

[The other two angles are also supplementary, since the sum of all the four angles = four rt angles. (I. 32)].

A circle may be described about ABCD

Describe a ⊙ about the △ ABD. [IV 5.

The ○ce of this ⊙ will pass through C

If not, let the ○ce cut BC or BC produced, at E

The ∠ DEB is supplementary to the ∠ BAD, [III 22.

also, the ∠ DCB is supplementary to the ∠ BAD [*Hyp.*

∴ the ∠ DEB = the ∠ DCB

which is absurd [I 16.

∴ the ⊙ will not cut BC in any other point but C.

∴ the ⊙ will pass through C

∴ the quadrilateral ABCD is cyclic.

Wherefore, *if any two opposite angles* &c Q. E. D.

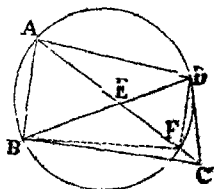
Proposition IV Theorem.

If the diagonals of a quadrilateral cut each other, in such a manner that the rectangle under the segments of the one be equal to the rectangle under the segments of the other, the quadrilateral is cyclic

Let ABCD be the quadrilateral, let its diagonals AC, BD, cut each other at E

If the rect. AE, EC = the rect. BE, ED:

then the quadrilateral ABCD *is cyclic.*

Describe the ⊙ ABD, about the △ ABD. [IV 5.

The Oce of this ⊙ will pass through C

If not, let the ⊙ cut AC at F.

The rect. BE, ED = the rect AE, EF. [III. 35.

Also, the rect BE, ED = the rect. AE, EC. [*Hyp.*

∴ the rect AE, EF = the rect AE, EC;

∴ EF = EC, which is impossible

∴ the ⊙ does not cut AC at any other point, but C

∴ the quadrilateral ABCD is cyclic.

Wherefore, *if the diagonals* &c Q E D

Proposition V Theorem.

If two opposite sides of a quadrilateral be produced to meet, and if the rectangle contained by the whole of a side produced and the part produced, be equal to the rectangle contained by the whole of the other side produced and the part produced, the quadrilateral is cyclic

Let the side AD and BC of the quadrilateral ABCD be produced to meet at E

if the rect AE, DE = the rect BE, CE

the quadrilateral ABCD is cyclic

Describe a ⊙ about the △ ABD [IV 5

It will cut BE at C

If not, let the ⊙ cut BE at F

The rect AE DE = the rect BE, FE [III 36 *Cor.*

Also, the rect AE, DE = the rect BE CE [*Hyp.*

∴ the rect BE, FE = the rect BE, CE

∴ FE = CE,

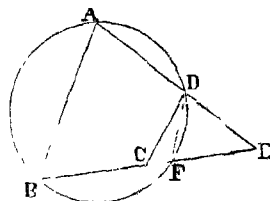
which is impossible,

∴ the ⊙ will not cut BE at F,

and we can shew that it will not cut BE in any other point but C.

∴ the ⊙ will pass through C

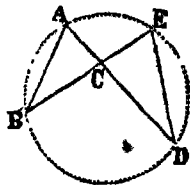
Wherefore, *the quadrilateral* ABCD *is cyclic.* Q. E. D.

Def. *In equiangular triangles, the sides opposite to the equal angles are called* ***Corresponding sides***

Proposition VI. Theorem.

In two equiangular triangles, the rectangles under the non-corresponding sides about any pair of equal angles are equal.

Let ABC, CDE, be two equiangular $\triangle$s;

the $\angle$ ABC = the $\angle$ CDE,
the $\angle$ BAC = the $\angle$ CED,
the $\angle$ ACB = the $\angle$ ECD

Place the $\triangle$s, so that the angular points of the pair of equal angles ACB, DCE, may be in one point, and that the side AC may be in the same straight line with CD, the angle opposite to which being not equal to the angle opposite to AC.

The $\angle$ ACB = the $\angle$ ECD

Add to each the $\angle$ BCD.

$\therefore$ the sum of the $\angle$s ACB, BCD = the sum of the $\angle$s BCD, DCE.

The $\angle$ ECD = the $\angle$ ACB

To each add the $\angle$ BCD

$\therefore$ the sum of the $\angle$s BCD, ECD = the sum of the $\angle$s ACB, BCD
= two right angles. [I 13.

$\therefore$ BC is in the same straight line with CE. [I 14.

Now, because the $\angle$ BAC = the $\angle$ CED,

$\therefore$ the points A, B, E, D are concyclic [Prop I.

$\therefore$ the rect. AC, CD = the rect BC, CE. [III 35.

If the $\triangle$s be similarly placed about the angles B, D, and about A, E, we can likewise prove that

the rect AB, CD = the rect. BC, DE,

and the rect. AB, CE = the rect AC, DE. Q. E. D.

Proposition VII (*Simpson's Theorem*)

If from any point on the circumference of a circle circumscribing a triangle, perpendiculars be dropped on its sides or sides produced, the feet of these perpendiculars are collinear; and if from a point the feet of the perpendiculars on the sides of a triangle be collinear, the point is concyclic with the angular points of the triangle.

Let ABC be a △.

(1) If D be a point on the ⊙ce of the circle described about the △, and DE, DF, DG, be perps. on BC, AC, AB, respectively:

the points E, F, G, *are collinear.*

(2) And if from any point D perps. DE, DF, DG, be dropped on the sides of the △ so that the feet of the perpendiculars be collinear

then A, B, C, D, *are concyclic*

Join DA, DC, GF, FE

(1) ∵ each of the ∠s AGD, AFD, is a right angle,

∴ the quadrilateral AFDG is cyclic, [Prop III.

∴ the ∠ DFG = the ∠ DAG, [III 21.
= the ∠ DCB [Prop. II

Again, ∵ the ∠ DFC = the ∠ DEC, being rt angles,

∴ CDFE is cyclic, [Prop I.

∴ the ∠ DFE = the supplement of the ∠ DCE [III. 22.
= the supplement of the ∠ DFG.

∴ GF, FE are in the same straight line. [I. 14.

(2) Let G, F, E, be collinear.

As in (1), the quadrilaterals CDFE, AFDG, are cyclic

∴ the ∠ EDC = the ∠ EFC, [III. 21.
= the ∠ AFG, [I 15.
= the ∠ ADG. [III 21.

To each add the ∠ ADE:

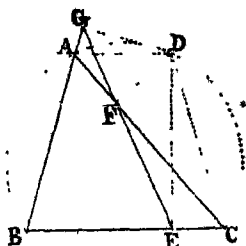
∴ the ∠ ADC = the ∠ GDE
= the supplement of the ∠ B,

since DEBG is cyclic [Prop. III.

∴ ABCD is cyclic [Prop III.

Q. E. D.

Def *The line of collinearity* EFG *is called* ***Simpson's line*** *(or the* ***pedal line****), of the point* D, *with reference to the triangle* ABC

NOTE. The Simpson's line of each angular point of the triangle is the altitude through that point, also, the Simpson's line of the other end of the diameter through an angular point is the base of the triangle opposite that angular point.

Proposition VIII Theorem

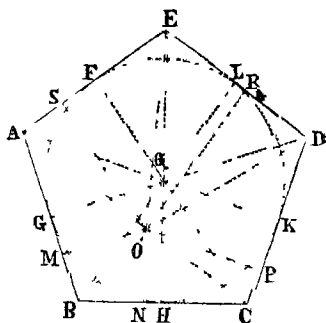
The sum of the perpendiculars on the sides of a regular polygon, of n *sides from any point within the figure, is equal to* n *times the radius of the inscribed circle*

Let ABCDE be a regular figure of *five* sides, GHKLF is the inscribed circle touching the sides of the figure ABCDE at G, H, K, L, F, O any point within the figure

Let OM, ON, OP, OR OS be perps to the sides of the polygon

The sum of the perpendiculars from O *on the sides of that polygon is equal to* five *times the radius of the inscribed circle*

Find Q the centre of the inscribed ⊙

Each of the straight lines joining Q and the points of contact G H, K, L, F is perp to the sides, and is also the radius of the inscribed circle

The areas of the five $\triangle$s formed by joining O with A, B C, D, E, =the areas of the five $\triangle$s formed by joining Q with A, B C D, E, since each of these groups of $\triangle$s make up the whole polygon

But double the areas of the first group of $\triangle$s
=the sum of the rects on each side and the perp on it from O,
=the rect on a side and the sum of the perps from O on the sides [II 1,
since the sides of the polygon are equal

And double the areas of the second group of $\triangle$s
=the sum of the rects on each side and the radius from Q,
=the rect on a side and the sum of the radii from Q, [II 1
since the sides of the polygon are equal

∴ the rect contained by a side and the sum of the perps from O
=the rect contained by a side and the sum of the radii from Q.
∴ the sum of the perps from O=the sum of the radii from Q

Likewise, we can prove the above when the figure is any regular polygon

Wherefore, *the sum of the perpendiculars*, &c Q. E D

OBS The sum of the perpendiculars from the points of contact on the sides of a regular polygon of *n* sides described about a circle is equal to *n* times the radius of the circle

Proposition IX. Theorem.

The locus of the centres of the circles inscribed in all right-angled triangles on the same hypotenuse is the arc of the quadrant described on the hypotenuse

Let AB be the hypotenuse, and let ACBO be a quadrant, *then the arc* ACB *is the locus of the centres of the* ⊙*s inscribed in all right-angled* △*s on the same hypotenuse* AB.

If the whole ⊙ be completed,

then the ∠ in the larger segment on AB = half the ∠ O, at the centre [III 20
= half a right angle

But the ∠ACB = the supplement of the ∠ in the larger segment on AB, [III 22

∴ the ∠ ACB = the supplement of half a right angle
= one and a half right angles

Take any point C on the arc AB

Draw CD ⊥ AB,

and with centre C and radius CD describe the ⊙ DEF

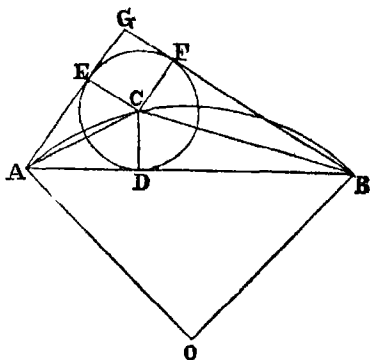
Draw AE, BF, tangents to this ⊙ and cutting each other at G.

Join CE, CF

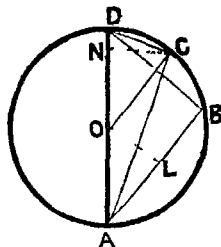
∴ the two △s ACD, ACE, are congruent, [I 8.
and the two △s BCD, BCF, are congruent.
∴ the ∠ ACD = the ∠ ACE,
and the ∠ BCD = the ∠ BCF
∴ the ∠ ACB = the sum of the ∠s ACE, BCF,
∴ the sum of the ∠s ACB, ACE, BCF = double the ∠ ACB
= three right angles.
∴ the remaining ∠ ECF = a right angle.
∴ the ∠ G = a right angle,
since the ∠s E and F are right angles.

∴ AGB is a △ right-angled at G, and C the centre of the inscribed ⊙ lies on the arc AB

Similarly, if we take the point C anywhere else on the arc AB, we may shew that it is the centre of another ⊙ inscribed in another right-angled △ whose hypotenuse is AB.

∴ the arc AB is the locus of the centres of ⊙s inscribed in all the right-angled △s on AB. Q. E. D.

Proposition X Problem.

To find the length of the circumference of a circle in terms of the diameter.

Let ABCD be a ⊙, AD being the diameter, and O the centre.

Let DC, CB, be any two equal arcs
Join AB, BD AC, CD, OC
Draw OL ⊥ AB, and CN ⊥ AD

Then, ∠ DOC = 2 ∠ DAC
= ∠ DAB, since arc DC = arc CB.
Hence, in the △s ONC, OLA,
OC = OA,
∠ CON = ∠ OAL,
and ∠ N = ∠ L, being right angles ·
∴ ON = AL
But AB = 2AL [III. 3.
∴ AB = 2ON
Now $AC^2 = OA^2 + OC^2 + 2ON \cdot OA$ [II 12.
$= 2OA^2 + AB \cdot OA$
Let the radius $= r$,
∴ $AC^2 = 2r^2 + r \cdot AB$ ... . (1)

But arc AB = the arc supplemental* to arc DB,
and arc AC = the arc supplemental to arc DC, i e to half the arc DB,
also, chord AB = the chord in the arc supplemental to arc DB,
and chord AC = the chord in the arc supplemental to half the arc DB.

Hence we see from (1) that

the square on the chord in the arc supplemental to half any given arc $= 2r^2 + r \times$ (the chord in the arc supplemental to the given arc).

* The arc supplemental to an arc is that arc which together with the first = a semi-circumference.

Now let the given arc DB $= \frac{1}{6}$ of the $\bigcirc^{ce}$,

$\therefore$ the chord DB $= r$. [IV. 15.

But $AB^2 = AD^2 - DB^2$, [I. 47.

$\therefore AB^2 = 4r^2 - r^2 = 3r^2$,

$\therefore AB = r\sqrt{3}$

$= r \times 1\ 7320508076$ .. . .(2)

Hence the chord in the arc supplemental to $\frac{1}{6}$ of the $\bigcirc^{ce}$

$= r \times 1\ 7320508076$.

But from (1) and (2) we get

$$AC^2 = 2r^2 + r^2 \times 1\ 7320508076$$

$$= r^2(2 + 1\ 7320508076)$$

$$= r^2(3{\cdot}7320508076)$$

$$\therefore AC = r\sqrt{(3\ 7320508076)} \quad .\ .(3)$$

Now we may bisect the arc DC at C_1, the arc DC_1 at C_2, the arc DC_2 at C_3, etc

Hence we see from (1) and (3) that

$$AC_1^2 = 2r^2 + r\ AC$$

$$= 2r^2 + r^2\sqrt{(3\ 7320508076)}$$

$$= r^2\sqrt{(7\ 7320508076)}$$

$$\therefore AC_1 = r\sqrt[4]{(7\ 7320508076)} \ldots\ldots\ \ldots\ldots\ .\ .(4)$$

Also, from (1) and (4) we get

$$AC_2^2 = 2r^2 + r\ AC_1$$

$$= 2r^2 + r^2\sqrt[4]{(7\ 7320508076)}$$

$$= r^2\sqrt[4]{(23\ 7320508076)}$$

$$\therefore AC_2 = r\sqrt[8]{(23\ 7320508076)}$$

. . . .

$$\therefore AC_7^2 = r^2 \times 3\ 9999832669\ ,$$

and arc $DC_7 = \left(\frac{1}{2^8} \times \frac{1}{6}\right)$th part of the $\bigcirc^{ce}$

$= \frac{1}{1536}$th part of the $\bigcirc^{ce}$.

Also, chord $DC_7^2 = AD^2 - AC_7^2$ [I. 47.

$= 4r^2 - r^2 \times 3.9999832669$

$= r^2 \times .0000167331.$

$\therefore\ DC_7 = r \times .0040906112$

Now $1536 \times DC_7$ = the perimeter of a regular polygon of 1536 sides inscribed in the ⊙

But this perimeter almost coincides with the ○ce;

$\therefore$ $1536 \times r(.0040906112)$ = the ○ce of the circle, approximately.

$\therefore$ the approximate ○ce of the circle $= 1536 \times r \times .0040906112,$

$= 2r \times 768 \times .0040906112,$

$= 2r \times 3.14159,$

$=$ diameter $\times 3.14159.$

Hence the circumference of a circle is approximately 3.14159 times the diameter

This multiplier 3.14159 is usually denoted by the Greek letter π

NOTE Of course the value of π would have been still more exact had we taken more than 1536 sides in the inscribed regular polygon However for all practical purposes the value of π correct to 5 decimal places is quite sufficient to give the circumference of a circle In fact it gives the circumference of the earth in terms of the diameter, correct to within a fraction of a mile We shall shew this as an interesting illustration —

The diameter of the earth = 7917.6 miles.

$\therefore$ the circumference $= 7917.6 \times 3.14159$ miles

$= 7917.6\left(3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000}\right)$ miles.

$\therefore$ the fifth decimal place gives

$7917.6 \times \frac{9}{100000}$ mile

or .712584 mile.

Hence the fifth decimal place only affects a fraction of a mile.

Proposition XI Problem

To find the area of a circle in terms of the radius.

Let AC be a side of a regular polygon of n sides inscribed in a circle. Let K be the centre of the circle. Join CK and AK. Let P denote the perimeter of the circle, and p that of the polygon; also let r denote the radius. By joining the angular points with the centre, the polygon may be divided into n triangles, the area of each of which is equal to that of ACK.

Area of the polygon $= n$ times area of ACK

$$= n \times \frac{AC \times \text{perpendicular}}{2}$$

$$= \frac{p \times \text{perpendicular}}{2} \qquad [\text{since } p = n \times AC$$

Now if n be assumed to be indefinitely great, then AC will be indefinitely small, and therefore the perimeter of the polygon will almost coincide with the circumference of the circle. Therefore the area of the polygon will almost be equal to the area of the circle. But when AC becomes indefinitely small, the perpendicular from K on AC becomes very nearly equal to the radius of the circle.

$\therefore$ ultimately the area of the polygon $= \frac{p \times r}{2}$.

But $p = P$, when n is indefinitely large.

$$\therefore \frac{p \times r}{2} = \frac{P \times r}{2}$$

Also, $P = 2r \times 3{\cdot}14159$ [Prop. X.

$\therefore$ the area of the polygon $= \frac{2r^2 \times 3{\cdot}14159}{2} = r^2 \times 3{\cdot}14159.$

But the area of the polygon is proved to be equal to that of the circle, when n is indefinitely large.

$\therefore$ the area of the circle $= r^2 \times 3{\cdot}14159.$

GEOMETRICAL EXERCISES ON BOOK IV.

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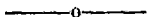
1. In a given circle, place a straight line equal and parallel to a straight line given in position, and not greater than the diameter.

2. In a given circle, inscribe an equilateral triangle.

3. A triangle is inscribed in one of two concentric circles, to inscribe in the other circle a triangle equiangular to the former, and so that its sides may be parallel to the sides of the other

4 Any number of triangles on the same side of the same base and having equal vertical angles will be circumscribed by one circle.

5 Two equilateral triangles are described about the same circle; show that their intersections will form a hexagon, equilateral, but not generally equiangular

6 The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle, inscribed in a circle whose diameter is the base

7 Inscribe a square in a given right angled isosceles triangle so that one of the sides of the square may coincide with the hypotenuse.

8 The centre of the circle inscribed in an equilateral triangle is the same with that of the circle circumscribed about the same triangle

9 In a given circle to inscribe a quadrilateral, two of whose opposite sides shall be at given distances from the centre of the circle and inclined at a given angle

10. Inscribe a circle in a quadrant

11. Inscribe a circle in a rhombus

12. Inscribe a regular hexagon in a given equilateral triangle

13 If a circle be inscribed in a right angled triangle, the difference between the hypotenuse and the sum of the other sides is equal to the diameter of the circle

14 In a triangle ABC, let AD bisect the angle A, and meet BC at D From O the centre of the inscribed circle, draw OE perpendicular to BC; then is the angle BOE equal to the angle DOC.

15. Draw from the obtuse angle of a triangle to the base a straight line, the square on which shall be equal to the rectangle contained by the segments of the base

16 Describe a circle which shall pass through one angle and touch two sides of a given square

17 The square on the side of an equilateral triangle inscribed in a circle is three times the square on the side of a hexagon inscribed in the same circle

18. Describe a circle which shall touch a given circle and two given straight lines which themselves touch the given circle.

19 Describe a circle about a given rectangle

20 Describe a circle which shall touch a given straight line, and pass through two given points

21 If two triangles ABC, DEF be inscribed in the same circle, so that AD, BE, CF intersect each other at one point O, prove that, if O be the centre of the inscribed circle of one of the triangles, it will be the point of intersection of the perpendicular in the other, drawn from the angular points on the sides

22 Show that the straight lines joining the centres of the circles touching one side of a triangle and the other sides produced, pass through the angular points

23 Describe a circle which shall pass through a given point and touch a given circle at a given point

24 The line joining the centres of the inscribed and circumscribed circles of a triangle, subtends at any one of the angular points an angle equal to the semi difference of the other two angles

25 Find the centre of a circle which shall cut off from the sides of a triangle three chords equal to one another

26 If a circle be inscribed in a triangle, the distance of any angle of the triangle from the point of contact of the circle with one of the sides which contain it, is equal to half the excess of the sum of these sides above the side opposite to the angle

27 If the circle inscribed in a triangle ABC touch the sides AB, AC at the points D, E, and a straight line be drawn from A to the centre of the circle meeting the circumference in G, show that G is the centre of the circle inscribed in the triangle ADE

28 Given the three angles of a triangle and the radius of the inscribed circle construct the triangle

29. In a given circle inscribe a rectangle equal to a given rectilineal figure

30 Describe a circle which shall touch a given circle in a given point, and also touch a given straight line

31 In a given circle inscribe four equal circles touching each other and the given circle.

32 Find the locus of the centres of the circles, inscribed in all triangles on the same base, and having equal vertical angles.

33 Describe a circle which shall touch a given circle and pass through two given points.

34. Describe a regular octagon on a given straight line

35 Describe a circle which shall pass through two given points, and cut off from a given straight line a chord equal to another given straight line.

36 Describe a circle touching two sides and passing through one angle of a given rhombus

37 Two straight lines meet at a point describe a circle which shall have its centre in a third straight line that does not meet the others at their point of intersection, and shall cut off equal chords from the first two straight lines

38 The straight lines joining the centres of the escribed circles with the opposite angular points of the triangle, intersect each other at the centre of the circle inscribed in the same triangle

39 Given the angles of a triangle, and the perpendiculars from any point on the three sides, construct the triangle

40 Inscribe an equilateral triangle in a square (1) when the vertex of the triangle is on one of the angles of the square, (2) when the vertex of the triangle is on the middle point of a side of the square

41 If the inscribed and circumscribed circles of a triangle have the same centre, show that the triangle is equilateral

42 A circle is described about an isosceles triangle, show that the straight line joining the centre of the circle with that of the inscribed circle shall pass, when produced, through the vertical angle of the triangle

43 From any point B in the radius CA of a given circle whose centre is C, a straight line is drawn at right angles to CA meeting the circumference at D, the circle described about the triangle CBD touches the given circle at D

44 Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point, not in the same straight line with the other two, may be equal to a given straight line

45 Having given one side of a triangle, and the centre of the circumscribed circle, determine the locus of the centre of the inscribed circle

46 The opposite sides AB, DC of a quadrilateral inscribed in a circle are produced to meet at E, and the sides AD and BC at F, circles are described about the triangles DCF and BCE cutting each other again at G, show that E, F, G are in one straight line

47 Describe six equal circles touching each other and a given circle, and show that each of the circles so described must be equal to the original circle.

48 Given the vertical angle of a triangle, and the radii of the inscribed and circumscribed circles construct the triangle

49 Given the base and the vertical angle of a triangle, and also the radius of the inscribed circle, construct the triangle.

50 Given the base and the vertical angle of a trangle find the locus of the point of intersection of the perpendiculars from the extremities of the base on the opposite sides.

51 The diameter of the circle inscribed in a right-angled triangle, is equal to the excess of the sum of the sides (which contain the right angle) above the hypotenuse

52 Having given the hypotenuse of a right angled triangle and the diameter of the inscribed circle, construct the triangle

53 In an isosceles triangle which has each of the angles at the base double the third angle, the difference of the squares on one side and the base, is equal to their rectangle

54 Inscribe a regular duo-decagon in a given circle, and show that its area is equal to that of a square on the side of an equilateral triangle inscribed in the same circle

55 In Fig Prop 10, if DA be produced to meet the circle in F, and FB be joined show that in the isosceles triangle thus formed, the vertical angle is eight times each of the angles at the base

56 Describe a circle which shall touch a given circle, and each of two given straight lines which meet each other

57 Two semicircles are described on the sides of a right-angled triangle describe a circle which shall touch them and shall have its centre in the hypotenuse

58 Inscribe three circles in an isosceles triangle touching each other, and each of them touching two of the three sides of a triangle.

59 Divide a circle into two parts, such that the angle contained in one segment, shall be equal to four times the angle contained in the other

60 Inscribe a square in the space included between two equal circles which cut each other

61 In fig Prop 10, show that the base of the triangle is equal to the side of a regular decagon inscribed in the larger circle.

62 Upon a given straight line as base, describe an isosceles triangle having the third angle three times each of the angles at the base

63 ABC is an isosceles triangle, of which the angles B and C are each double of A, prove that the square on AC is equal to the square on BC together with the rectangle contained by AC and BC.

64 Describe a circle about a figure formed by constructing an equilateral triangle upon the base of an isosceles triangle, the vertical angle of which is four times each of the angles at the base.

65 Describe a regular pentagon about a circle without first inscribing one

66 If the two diagonals of a regular pentagon be drawn to cut one another, the greater segment will be equal to the side of the pentagon

67 Each of the triangles made by joining the extremities of the adjacent sides of a regular pentagon, is less than a third and greater than a fourth of the whole area of the pentagon

68 In fig Euc. IV 10, the squares on AB, AC are together equal to the square on the side of a regular pentagon inscribed in the circle BDE

69 The square described on the side of a regular pentagon in a circle, is equal to the square on the side of a regular hexagon, together with the square on the side of a regular decagon in the same circle.

70 Show how to derive the hexagon from an equilateral triangle inscribed in a circle, and from this construction show that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle

71 In fig Euc IV 10, if A be the vertex, and BD the base of the constructed triangle, D being one of the points of intersection of the two circles employed in the construction, and E the other, and AE be drawn meeting BD produced in F, prove that FAB is another isosceles triangle of the same kind

72 If a polygon of any odd number of sides have all its angular points on the same circle, and all its angles equal, then shall its sides be equal

73 Show that if the circumference of a circle passes through three angular points of a regular polygon, it will pass through all of them.

74 Inscribe a square in a semicircle

75 In a given circle inscribe a triangle whose angles are as the numbers 2, 5 and 8

76 Prove that the smaller of the two circles, employed in the construction of Prop 10, is equal to the circle described about the required triangle

77 In a triangle, the centre of the inscribed circle, the centre of any escribed circle and the ends of the side which the escribed circle touches, are concyclic

78 Of the four points, the in-centre and the three ex-centres of a triangle, any one is the ortho centre of the triangle formed by the other three points

79 In any triangle, if a, b, c, represent the sides, and s represent half their sum, show that the area of the triangle $=\sqrt{\{(s-a)(s-b)(s-c)s\}}$

80 In any triangle if r represent the radius of the inscribed circle, and r_1, r_2, r_3 represent the radii of the three escribed circles, show that the area of the triangle$=\sqrt{(r\ r_1,\ r_2\ r_3)}$

81 Given the base, the vertical angle, and the radius of any of the escribed circles, construct the triangle

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HINTS FOR SOLUTION.

BOOK IV.

Prop 1

Let A be the given point and C the centre of the given circle Describe a circle with AC as diameter, and place in it the straight line AB equal to the given distance Join BC, cutting the circle at D, E DE is the required diameter

Prop 2

From the centre draw perpendiculars to the sides, etc

Prop 3

1 The figure is divided into four equal equilateral triangles

2 In fig Euc IV 4, join AD The area of the triangle ABC is equal to the areas of the triangles ADB, BDC CDA But the rectangle contained by AB, ED is double of the triangle ABD, etc

Prop 4

6 Let AC cut the parallel straight lines AB, CD

Let AE, CE bisect the angles BAC, ACD, E is the centre of the required circle

Prop 5

2 The circles are concentric

Prop 9

Let AC, BD, the diagonals of the rectangle, bisect each other at O. O is the centre of the required circle

Prop 10.

1. The triangle ABD may be inscribed in the circle ACD

2. On the given straight line BD describe an isosceles triangle ABD equiangular to the triangle constructed in Prop 10. Describe a circle with A as centre and AB or AD as radius Produce BA DA to meet the circle at E, F Bisect the angle BAF by AG and bisect again GAF, BAG, and produce the bisecting lines. The ten angles at A are equal, etc.

3 Angle AFD=angle BCD=angle ABD

$\angle ADF = \angle AFD = \angle ABD = \angle ADB.$

Also AD is common to the two Δs AFD, ABD.

$\therefore \Delta AFD = \Delta ABD$ [I 26

4 Let ABC be a right angle On BC describe the isosceles triangle CBD, so that the angle CBD may be double the angle CDB

$\angle CBD = \frac{4}{5}$ of a right angle Bisect the angle CBD, &c.

Prop 11

1 On the given straight line AB describe an isosceles triangle ABD equiangular to the triangle constructed in Prop 10 Describe a circle about ABD, bisect the arc AD, BD at E, F, $ABFDE$ is the required pentagon

2 In the fig, $\angle ABC = \frac{6}{5}$ of a right angle (I 32, *Cor* 1).

$\therefore \angle BAC = \frac{2}{5}$ of a right angle

$\therefore \angle EAC = \frac{4}{5}$ of a right angle, and $\angle AED = \frac{6}{5}$ of a right angle.

$\therefore$ AC is parallel to ED

3 $\angle AEB = \angle EAD = \frac{2}{5}$ of a right angle

$\therefore \angle AKB = \frac{4}{5}$ of a right angle, also $\angle BAK = \frac{4}{5}$ of a right angle

$\therefore AB = BK = BC = CD = DE = DK$

4 $BKD = \frac{6}{5}$ of a right angle, &c

Prop 15

2 On the given straight line AB, describe an equilateral triangle ABC With C as centre and CB or CA as radius, describe a circle $ABDE$ Draw the diameters AD, BE Bisect the angle ACE, etc

3 Let C be the centre of the circle Draw any diameter ACB. At C in AC make the angle $ACD = \frac{2}{3}$ of a right angle, also make the angle $DCE = \frac{2}{3}$ of a right angle Produce DC, EC to meet the circumference at F and G Tangents at A, G, F, B, E, D will form the required hexagon

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Miscellaneous Exercises on Book IV

Hints for Solution

1. Find the centre D Through D draw the diameter BDC parallel to the given straight line A From DB, DC cut off DE, DF each equal to half of A From E, F, draw FG, EH, at right angles to BC, meeting the circle at G, H Join GH GH is the required line

2 This is only a case of Prop 2 of Euc IV, the given triangle being equilateral

3 Join the centre with the angular points and let these lines, produced if necessary, cut the other circle at D, E, F, DEF is the required triangle

4 Apply Euc III 21

5 If the three sides of the second equilateral triangle touch the middle points of the three equal arcs, into which the circle is divided by the points of contact of the first triangle, then the hexagon becomes equilateral as well as equiangular, otherwise it is not equiangular

6. Let ABC be an equilateral triangle On AB as diameter describe the circle $ALDB$ cutting BC at D Join AD ADB is a right angle (III 31) From D draw DO perpendicular to AB. Produce DO to meet the circle again at E Join AE

7 Let ABC be the right angled isosceles triangle the angle at A being a right angle From A draw AD perpendicular to BC Trisect AD at E and O Through E draw FEK parallel to BC, meeting AB, AC at F, K respectively Join FO, KO, and produce them to meet BC at H, G Join FG, KH $FGHK$ is the required square

8 The points in which the perpendiculars from the point of intersection of the lines bisecting the angles meet the sides, are also the middle points of the sides (Euc IV 4, 5)

9 Draw two radii OA, OB making an angle between them equal to the supplement of the given angle From OA, OB cut off OC, OD equal to the given distances From C, D, draw CE, DE, at right angles to OA, OB, respectively and meeting each other at E. Let CE cut the circle at F, G, and DE at H, K. Join FH, GK. $FHKG$ is the required quadrilateral

10 Let OAB be a quadrant, AOB being the right angle Bisect the arc AB at C Complete the circle Through C draw a straight line touching the circle and meeting OA, OB produced, at D, E respectively Then apply Euc IV 4. The construction holds good for inscribing a circle in any sector.

11 Let $ABCD$ be the given rhombus. Draw the diagonals AC, BD cutting each other at O From O draw perpendiculars to the sides, etc

12 Trisect the sides and join the points of trisection adjacent to the angles, etc.

13 The segments of the sides containing the angles are equal.

14. Apply Euc I 32

15 Describe a semicircle on the line joining the obtuse-angle with the circum-centre. The line joining the obtuse-angle to the point of intersection of the semicircle and the base, is the required line.

17 In Fig Euc IV 15, join AE, EC, AC. AEC is an equilateral triangle Let EC cut DG at O DO is equal to OG, and EO is at right angles to DG Apply Euc II 12

18 Let AB and CD, the two given straight lines, touch the given circle at A, C They may meet or may not meet Bisect the arc AC at E. Find the centre P Join EP Through E, draw the tangent GEH meeting AB at G and CD at H Bisect the angle BGE by GF, meeting PE produced at F F is the centre and FE is the radius of the required circle

19 The point of intersection of the diagonals is the centre of the required circle

20 Let A, B be the two given points and CD the given straight line Join AB and produce it to meet CD at D. From DC cut off DE so that the rectangle contained by AD, BD may be equal to the square on DE (II 14) Describe a circle about the triangle ABE.

21. Apply Euc III 21

22 The straight line joining the point of intersection of two tangents to a circle with its centre, bisects the angle made by the tangents

23 Let C be the given point, and A the given point in the circumference of the given circle Find its centre B. Join BA and produce it to D, and join AC At the point C in AC make the angle ACD equal to the angle CAD D is the centre of the required circle.

24 Apply Euc. IV. 4, 5

25 The centre of the inscribed circle is the required centre.

26. The two tangents to a circle which may be drawn from any point without it are equal.

27 Let DE meet AO, the straight line from A to the centre O, in Q, AQO is perpendicular to DE $GD = GE$, by III 32, GD, DE bisect the $\angle$s ADE, AED.

28. Euc IV. 3

29. Draw AB a diameter of the given circle, and on it describe a parallelogram equal to the given rectilineal figure (I 44), the side opposite the diameter cutting the circle at C Join AC, BC Draw BD parallel to AC Join DA $ACBD$ is the required rectangle.

30 Let BC be the given straight line, and A the given point in the circumference of the given circle At A draw a tangent to the circle meeting BC at B Bisect the angle ABC by BD Find the centre E join EA and produce EA to meet BD at D D is the centre of the required circle

31 In fig Euc IV 7, join GE, HE, KE, FE Inscribe circles in the triangles GEH, HEK, KEF and FEG

32 On the given base AB describe the segment ACB, containing an angle equal to the given vertical angle On AB describe another segment AOB, containing an angle equal to the sum of half the vertical angle and a right angle The arc AOB is the required locus From O as centre, and with OD perpendicular to AB, as radius describe the circle DEF Draw AF, BE touching the circle DEF at F, E AF, BE produced, shall meet in the circumference of ACB

33 Let ABC be the circle, and D E the given points take any point C in the circumference of the given circle, about CDE describe the circle $CBDE$ cutting the circle ABC at B again Produce CB, ED to meet at F, draw FG touching the circle ABC at G The circle described about GDE is the required circle

34 Let AB be the given straight line. At A, B in AB make the angles BAO, ABO each equal to three-fourths of a right angle. AO is equal to BO From the centre O at the distance OA or OB describe a circle Produce AO, BO to meet the circle at D, C; through O draw two diameters at right angles to OD, OC Join the points of intersection

35 Let E, F be the two given points, and CD the straight line from which the chord to be cut off, shall be equal to the given line AL If EF be parallel to CD, bisect EF at right angles by GH meeting CD at H On both sides of H cut off from CD the straight lines HM, HR each equal to half of AL Join EM, and bisect it at right angles by KN meeting GH at N N is the centre of the required circle If EF be not parallel to CD, let EF meet CD at O. Produce AL to K so that the rectangle contained by AK, KL may be equal to the square which may be made equal to the rectangle EO, OF (Ex. 13, page 193) From OC cut off OR, OM equal to AK, KL. The circle described about MFE is the required circle.

36. Let $ABCD$ be the rhombus. Draw AC and through C draw ECF at right angles to AC meeting AB, AD produced at E, F. Inscribe a circle in the triangle AEF

37 Let AC, AD be the two straight lines which meet at A, and EF the third straight line Bisect the angle CAD by AG meeting EF at G From G draw GH, GK perpendiculars to AC, AD From the centre G and with radius greater than GH describe any circle cutting AC and AD, it shall cut off equal chords

38 Apply Euc IV 4.

39 At any point O place the two given perpendiculars AO, DO making an angle between them equal to the supplement of one of the given angles, also place OC the third perpendicular, making the angle AOC equal to the supplement of another given angle; then DOC will be equal to the supplement of the third angle Through the points A, C, D draw EAF, FCG, GDE at right angles to AO, CO, DO respectively

40 (1) Let $ACDE$ be the given square Join AD At A in AD make the angles DAG, DAF each equal to one third of a right angle, meeting ED at G, and CD at F respectively Join GF (2) Bisect AC at H At H make the angles AHG, CHF, each equal to two-thirds of a right angle Join GF.

41 & 42 Apply Euc IV 4, 5

43 DC may be proved to be a diameter of the circle DBC

44 Let A, B be the two given points, C the other given point, and L the given straight line Draw CD perpendicular to AB, from CD or CD produced, cut off CE equal to L Join BE Draw EK at right angles to CE At B in EB, make the angle EBK equal to the angle BEK BK is equal to KE From the centre K, and with the radius KE or KB describe the circle MBE Let CB or CB produced, meet the circle at M The circle described about ABM is the required circle

45 Let A be the centre of the circumscribed circle, and CD the given side, the perpendicular from A on CD shall bisect it From A as centre and with AC or AD as radius, describe the circle CDE Then the segment CED contains the vertical angle, which is thus known Now proceed as in Ex 32, and Prop 3, page 353, of which the present is a particular case

46 Apply Euc III 22

47 Inscribe an equilateral and equiangular hexagon $ABCDEF$ in the given circle (see fig IV 15) Produce GA, GB, GC, etc, to K, L, M, etc, making each of AK, BL, CM, etc, equal to AG the radius of the circle From the centres K, L, M etc, and with radii equal to KA, LB, MC, etc, describe six circles.

48 Describe a circle with the given radius of the circumscribed circle In this circle draw BC, cutting off a segment containing an angle equal to the vertical angle (Euc III, 34) Now the exercise becomes almost the same as Ex 45

49 This problem is to be solved in the same manner as the preceding one

50 On the given base BC describe the segment of a circle BAC containing an angle equal to the supplement of the vertical angle The arc BAC is the required locus [*Prop G* Cor, p 379]

51 The radii drawn to the points of contact of the sides make a square with the parts of the sides between the right angle and the points

52 Let AB be the given hypotenuse On AB describe a semicircle ACB, and the segment of a circle ADB containing an angle equal to half a rt angle In the segment ADB place AD equal to the sum of AB and the given diameter of the inscribed circle, cutting ACB at C Join CB ACB is the required triangle

53 Apply Euc IV 10 and II 2

54 Inscribe an equilateral and equiangular hexagon in the given circle and then bisect the arcs, etc

55 Apply Euc I 32

56 Let AB AC be the two given straight lines and RQ the given circle Take O the centre of the circle Remote from O and at a distance equal to the radius of the circle, draw two straight lines DF, DF, parallel to AB, AC The centre of the circle which would touch DF and DE, and pass through O, is the centre of the required circle

57 Let ABC be the right-angled triangle, the angle at B being a right angle, AGB the semicircle on AB, and BHC on BC. Take D, E the centres of the circles AGB and BHC Bisect AC at F Join DF, FF Produce FD, FE to meet the circles at G, H respectively F is the centre and FG or FH is the radius of the required circle

58 Let ABC be the isosceles triangle, from the vertex A draw AD perpendicular to BC Inscribe circles in the triangles ABD, ACD Describe a circle touching AB, AC and these two circles (see Ex 56).

59. In fig Euc IV 11, CAD and CD are the two required segments.

60 Let A, B be the centres of the circles Bisect AB at O. At O in BO make the angle BOE equal to half a right angle, meeting one of the arcs at E Draw a chord ELK at right angles to AB Join KO Produce EO, KO to meet the other arcs at H, M. Join EM, MH and HK

61 Proceed as in Ex 2 Prop 10

62 At A, B the extremities of the base AB make the angles BAD, ABD each equal to two fifths of a right angle or equal to the vertical angle BAD in fig Euc IV 10

63 Divide AB at D so that the rectangle contained by AB, BD may be equal to the square on AD (II 11) Apply Euc II 2

64 Let ABC be the isosceles triangle, A being the vertex Let BDC be the equilateral triangle Prove that each of the angles ABD, ACD is a right angle Bisect AD at E E is the centre of the required circle

65 Find the centre O Draw any radius OA Draw the tangent AC at A At O in AO make the angle AOC equal to two-fifths of a right angle (IV 10) cutting AC at C Draw CDE another tangent, making DE equal to CD, &c

66 Apply Euc I 32, and III 21 and 22

67 In fig Euc IV 11 the triangle BCD is equal to AED, but less than ABD and greater than half of ABD

68 From the arc BF cut off BF equal to BD Join FD FD shall be the side of a regular pentagon inscribed in FBD Let FD cut AB at O AB bisects FD at right angles Produce BA to meet the circle at G BO is equal to OC Apply Euc II 13 and III 35

69 In the construction of Ex 68, FD is the side of a regular pentagon, BD or AC that of a decagon, and AB, or the radius of the circle, that of a regular hexagon

70 Bisect the arcs intercepted by the angular points of the equilateral triangle, and join the points of bisection with the angular points

71 AED is another triangle equal in every respect to ABD. Therefore the angle BAE is double of BAD

72 Apply Euc III 26, 29

73 The lines bisecting the angles shall meet at one point, which is the centre of the circle described about the three points.

74 Let AEB be the semicircle on AB as diameter Find the centre C From A draw AD at right angles to AB and make it equal to AB Join DC cutting the circle at E Draw EF perpendicular to AB From CB cut off CR equal to CF At C in RC make the angle RCS equal to FCE Join SE, SR $EFRS$ is the inscribed square Produce CA to G making GA equal to EF Produce DA to H making AH equal to FC Join GH A circle may be described about $HGDC$ Therefore the rectangle contained by DA, AH is equal to the rectangle contained by GA, AC But DA is double of AC Therefore GA is double of AH Hence EF is double of FC and equal to FR

75 Let BC be the side of a regular quindecagon inscribed in a circle If A be the centre of the circle, the angle BAC is four fifteenths of a right angle At B in AB make the angle ABD equal to two thirds of a right angle cutting AC at D The remaining angle ADB must be sixteen fifteenths of a right angle In the given circle inscribe a triangle equiangular to ABD

76 Let F be the point at which the circles cut again Join AF, FD ADF is equal to ABD in all respects

77 Let D be the centre of the circle inscribed in the triangle ABC, and E the centre of the escribed circle which touch BC. The angle DBE is a right angle and the angle DCE is a right angle. Apply Addl Prop III p 348

78 See Miscell Prop I p 384

79 See Notes Book II Prop 13, p 183

80 See fig Ex 78 Let r_1, r_2, r_3, be the radii of escribed ⨀s which touch a, b, c respectively, and let r be the radius of the in-circle

$$\text{Area of } \Delta = rs \qquad (\textit{See} \text{ Notes IV } 4, \text{ p } 341).$$

$$\text{Area of } \Delta = r_1(s-a) \qquad (\textit{See} \text{ Prop. } I \text{ p } 384).$$

$$= r_2(s-b)$$

$$= r_3(s-c)$$

$$\therefore (\text{Area})^4 = r\, r_1\, r_2\, r_3 s(s-a)(s-b)(s-c).$$

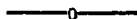
$$\therefore \text{Area}^2 = r\, r_1\, r_2\, r_3\,, \text{ \&c.}$$

81. Let AB be the base, CDE = vertical angle and R the radius of the circle touching the base AB. On AB describe a segment of

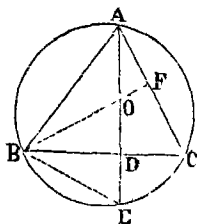
a circle containing an angle=half of the angle CDF. Draw $BG \perp AB$ and make BG=given radius. Draw $GH \parallel AB$. Join AH. Draw $HK \perp AB$ produced. With H as centre and HK as radius describe a circle. From B and A draw tangents to the circle, meeting at L. ABL is the required triangle.

The angle LBK=angles ALB, LAB.

$$\therefore HBK = \tfrac{1}{2}(ALB + LAB)$$

$$\therefore AHB = \tfrac{1}{2}ALB = \tfrac{1}{2}CDE, \text{ \&c}$$

ADDITIONAL MISCELLANEOUS PROPOSITIONS.

Proposition A Theorem

If the altitude of a triangle be produced to meet the circumference of the circle described about the triangle, the part of the altitude produced between the ortho-centre and the circumference is bisected by the base

Let ABC be a △, AD the altitude is produced to meet the ○ce at E

Draw BF ⊥ AC, cutting AD at O.

O is the ortho centre [Prop XVIII p 103.

Then OE *is bisected at* D

Join BE

The ∠ EBC=the ∠ EAC [III 21.
=the complement of the ∠ ACD
=the ∠ FBC

Hence in the △s BOD, BED,

the ∠ DBO=the ∠ DBE,

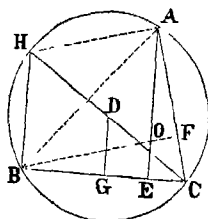
the ∠ BDO=the ∠ BDE,

and BD is common :

∴ DO=DE [I. 26.

Wherefore, *if the altitude* &c. Q. E. D.

Proposition B. Theorem

The straight line joining the ortho-centre of a triangle with any angle, is double the distance of the circum-centre from the opposite side.

Let O be the ortho-centre, and D the circum-centre of the △ ABC Draw DG ⊥ BC

Then AO *is double of* DG

Produce AO to meet BC at E produce BO to meet AC at F, and produce CD to meet the ○ce of the circum-circle at H Join AH, BH.

∵ HD=DC

and BG=GC · [III 3.

∴ DG=half of HB [Prop II p 90.

Again, HBC=a right angle, [III 31

∴ HB and AE are each perpendicular to BC,

∴ HB is parallel to AO

Also, HAC is a right angle, [III 31.

∴ HA and BF are each perpendicular to AC,

∴ HA is parallel to BO

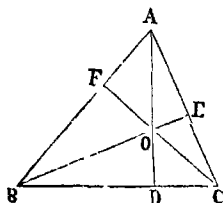
∴ HBOA is a parallelogram,

∴ AO=HB

=double of DG.

Wherefore, *the straight line* &c. Q. E D.

Proposition C Theorem.

The rectangles contained by the segments of the altitudes of a triangle are equal

Let AD, BE, CF, be the altitudes of the △ ABC,

and let O be the ortho-centre

Because the ∠ BFC=the ∠ BEC,

∴ the points B F, E, C, are concyclic [Prop. I p 347.

∴ the rect BO OE=the rect FO OC [III 35.

Likewise, C, D, F, A, are concyclic

∴ the rect FO OC=the rect AO OD

∴ BO OE=FO OC=AO OD

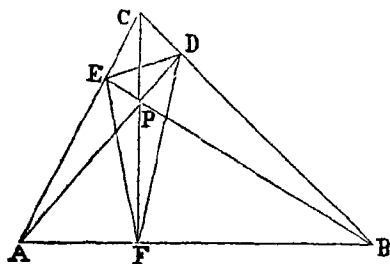
Wherefore, *the rectangles contained by &c* Q E D

Def 1 *In any triangle, the circle with the ortho centre as centre and radius equal to the straight line the square on which is equal to any rectangle under the segments of an altitude, is called the* ***polar circle*** *of the triangle*

Def 2 *The triangle formed by joining the feet of the altitudes is called the* ***pedal triangle*** *of the original triangle.*

Pedal triangle.

Proposition D Theorem.

The two adjacent sides of a pedal triangle make equal angles with that altitude at the foot of which they meet

Let AD BE, CF, be the three altitudes of the △ ABC DEF is the pedal △ of ABC

Then FD, ED,* *make equal angles with* AD , DE, FE, *make equal angles with* BE , *and* EF, DF, *make equal angles with* CF.

Let P be the ortho-centre

Because the ∠s PDB, PFB, are right angles,

∴ the points F, B, D, P, are concyclic [Prop III p 348

∴ the ∠ FDP=the ∠ FBP [III 21

=the complement of the ∠ BAC

Similarly P, D, C, E, are concyclic

∴ the ∠ PDE=the ∠ PCE [III 21.

=the complement of the ∠ BAC.

∴ the ∠ FDP=the ∠ PDE

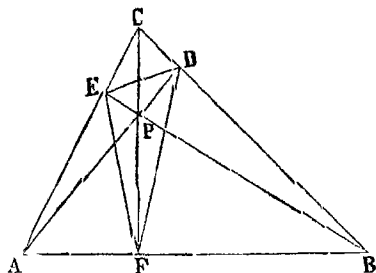
Likewise, the ∠ DEP=the ∠ FEP,

and the ∠ EFP=the ∠ DFP.

Wherefore, *the two adjacent sides* &c. Q E D

OBS. Since the internal bisectors of the angles of the triangle DEF meet at P, it follows that P is the in-centre of DEF Hence we have :—*The ortho-centre of a triangle is the in-centre of its pedal triangle.*

Proposition E Theorem.

In every triangle, each angle of the pedal triangle is supplementary to twice the angle of the triangle opposite the same

Let ABC be a triangle, DEF is its pedal triangle, P the ortho-centre

The angle EDF *is supplementary to twice the angle* BAC, &c.

Because the ∠ AEB = the ∠ ADB,

∴ A, B D, E are concyclic [Prop I p 347.

∴ the ∠ CDE = the ∠ BAC [Prop II p 347.

But the ∠ CDE = the ∠ BDF, [Prop *D*.

∴ each of the ∠s CDE, BDF = the angle BAC

∴ the ∠ EDF is supplementary to twice the ∠ BAC.

Likewise, each of the ∠s DEF, EFD, is supplementary to twice the ∠s ABC, BCA, respectively

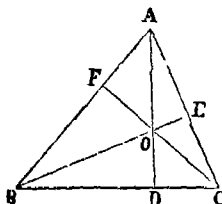
Wherefore, *in every triangle, each angle* &c. Q E D.

Cor Each of the triangles CED, AEF, DBF, is equiangular to the triangle ABC

Obs 1 If a circle be described about ABC, then EF is parallel to the tangent at A, FD parallel to the tangent at B, and ED parallel to the tangent at C.

Obs. 2. A, B, C, are the excentres of the pedal △ DEF.

Proposition F. Theorem

In every triangle each angular point is the ortho-centre of the triangle formed by the other two angular points and the ortho-centre of the original triangle, and the angles, subtended at the base by an angular point and by the ortho-centre, are supplementary

Let O be the ortho-centre of the △ ABC.

Then (1) A is the ortho-centre of the △ BOC,
B is the ortho-centre of the △ AOC,
C is the ortho-centre of the △ AOB,

and (2) the ∠s BAC, BOC, are supplementary,
the ∠s ABC, AOC, are supplementary,
the ∠s ACB, AOB, are supplementary

(1) ∵ CEA is perp to BO (produced),
and BFA is perp to CO (produced),
also AOD is perp to BC

∴ A is the ortho-centre of the △ BOC.

Similarly B, C, are the ortho-centres of the △s AOC, AOB, respectively

(2) ∵ the ∠s at E and F are right angles,

∴ A, E, O, F, are concyclic, [Prop III p 348.

∴ the ∠ BAC = supplement of the ∠ FOE [III 22.

= supplement of the ∠ BOC

∴ the ∠s BAC, BOC, are supplementary, whether we consider O as the ortho-centre of the △ BAC, or A as the ortho-centre of the △ BOC,

and similarly for the other angles

Wherefore, *in every triangle*, &c. Q E D

Cor DEF is the pedal triangle of all the four △s ABC, AOB, BOC, COA

Proposition G Theorem

In every triangle the circumscribing circle is equal to the circle described about the ortho-centre and any two angular points

(*See figure of the last Prop*)

We have to shew that

the ⊙ BAC=the ⊙ BOC=the ⊙ COA=the ⊙ AOB,

whether we consider O to be the ortho-centre of the △ BAC, or A, B, C, to be the ortho-centres of the △s BOC, COA, AOB, respectively

In the ⊙ BOC,

the ∠ in the segment conjugate to the segment BOC

=supplement of the ∠ BOC [III 22

=the ∠ BAC [Prop *F*

∴ the segment conjugate to BOC in the ⊙ BOC,

=the segment BAC of the ⊙ BAC [III 24.

∴ the whole ⊙ BAC=the whole ⊙ BOC

Similarly, we may shew that the ⊙ BAC=the ⊙ COA=the ⊙ AOB .

Wherefore, *in every triangle*, &c Q E D

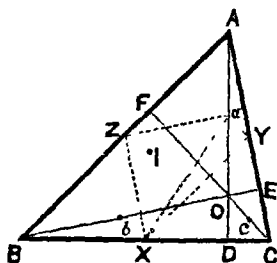
COR From this it is manifest that if A move along the arc BAC, O will move along the arc BOC, and *vice versa*, hence, *in every triangle if the locus of an angular point be the arc of a segment on the base, the locus of the ortho centre is the arc of a segment on the same base and on the same side of the base, and containing an angle supplementary to the angle in the first segment , and vice versa*

Proposition H. Theorem

Nine-Point Circle.

In every triangle, the feet of the altitudes, the middle points of the sides and the middle points of the straight lines joining the ortho-centre with each angular point, are concyclic.

Let D, E, F, be the feet of the altitudes of the △ ABC, X, Y, Z, the middle points of the sides, O the ortho-centre, and *a*, *b*, *c*, the middle points of OA, OB, OC, respectively

Then D, E, F, X, Y, Z, *a*, *b*, *c*, are concyclic.

Join XY, X*a*, XZ, Z*a*, *a*Y

∵ AB, AO, are bisected at Z, *a*,

∴ Z*a* is parallel to BOE [Prop II p 90.

Similarly ZX is parallel to AE ·

∴ the ∠ XZ*a*=the ∠ E=a right angle

Likewise, the ∠ XY*a*=a right angle,

also, the ∠ XD*a*=a right angle [*Constr.*

∴ the ⊙ drawn on X*a* as diameter passes through Z, Y, D

∴ X, D, Y, *a*, Z, are concyclic,

∴ the ⊙ XYZ (that is, a fixed ⊙, ∵ X, Y, Z, are fixed) passes through *a* and D

Similarly, the ⊙ XYZ passes through *b* and E;

likewise, the ⊙ XYZ passes through *c* and F

∴ the points D, E, F, X, Y, Z, *a*, *b*, *c*, are concyclic. Q. E. D.

Def. *Because this circle passes through nine particular points with reference to the triangle, therefore it is called the* **nine-point circle** *of the triangle.*

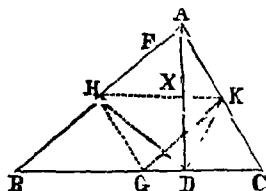
OBS. Since it passes through D, E, F, it circumscribes the pedal triangle of ABC

NOTE Since Xa, Yb, Zc, are diameters of the nine-point circle, therefore the centre lies at the middle point of each of these, hence these lines are concurrent at their common middle point which is the centre of the nine point ⊙; also, the radius of this ⊙ is equal to half of these lines

Moreover if I be the circum-centre of the △ ABC, then XI is perp to BC, that is, parallel to AO and also half of it (Prop *B*) Therefore XI is equal and parallel to Oa Therefore IXOa is a parallelogram (I 33), therefore its diagonals Xa, OI, bisect each other, that is the middle point of Xa is also the middle point of OI, therefore *the centre of the nine point ⊙ lies at the middle point of the line joining the ortho centre with the circum centre.*

Finally, since XI is equal and parallel to aA, therefore IA is also equal and parallel to Xa, that is, the radius of the circum-circle of ABC is equal to the diameter Xa of the nine point ⊙, therefore *the radius of the nine point circle is equal to half the radius of the circumcircle*

ALTERNATIVE PROOF *

In any triangle, the circle through the middle point of the sides (1) passes through the feet of the perpendiculars and also (2) passes through the middle points of the lines joining the orthocentre with the angular points of the triangle

Let ABC be any △, D, E, F, be the feet of the perps from A, B, C, respectively, G, H, K, the middle points of the sides Let AD cut KH at X

(1) *The circle passing through* K, G, H, *passes through* D, E, F.

∵ AH=HB, and AK=KC, ∴ HK is parallel to BC.

∵ AH=HB, and HX is parallel to BC ∴ AX=XD.

The ∠ AXH is a right angle, for it is equal to the ∠ ADB.

∴ in the △s AHX, DHX,

AX=DX,

HX is common,

and the ∠ AXH=the ∠ DXH

∴ the ∠ HAX=the ∠ HDX [I 4.

Similarly, the ∠ KAX=the ∠ KDX

∴ the whole ∠ KDH=the whole ∠ KAH.

Again ∵ AKGH is a parallelogram,

---

* I received this proof from Professor Nash, Inspector of European Schools, Bengal

∴ the ∠ KGH = the ∠ KAH [I 34.

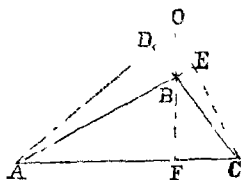
∴ the ∠ KDH = the ∠ KGH

∴ the ⊙ which passes through K, G, H, passes through D. [Prop. I p 347

Similarly, the same circle passes through E, F

Wherefore, *the circle which passes through the mid points of the sides passes through the feet of the perpendiculars*

CONVERSELY, the circle which passes through the feet of the perpendiculars, passes through the mid points of the sides For, only one circumference can pass through D, E, F [III 10.

(2) *The same circle bisects the lines joining the ortho-centre with the angular points of the triangle*

Let B be the orthocentre of the △ AOC , D, E, F, the feet of the altitudes

∵ ABC is a △ of which D, E, F, are the feet of the perps ,

∴ by the converse of (1), the ⊙ through D, E, F, passes through the middle points of BA, BC.

Likewise, since D, E, F, are the feet of the perps of the △ CBO,

∴ the same ⊙ passes through the middle points of BC, BO

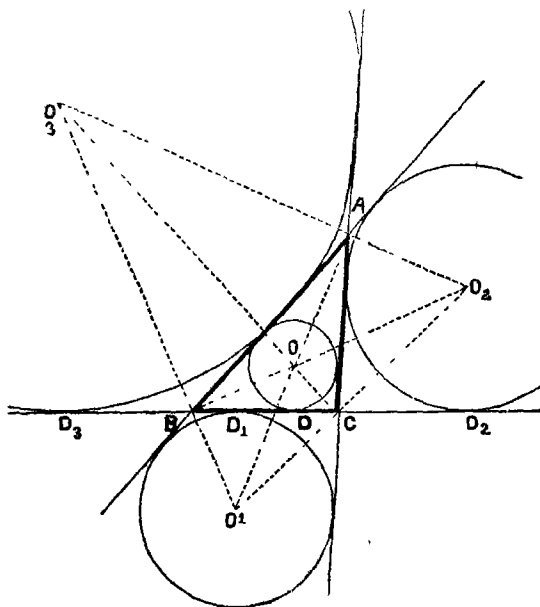
Hence the ⊙ passing through D, E, F, also passes through the middle points of BA, BC, BO

Wherefore, *in any triangle*, &c

Proposition I Theorem.

To describe four circles touching the sides of a given triangle.

Let ABC be a △.

Let OA, OB, OC, be the internal bisectors of the ∠s of the △

Then O is the in-centre of the △ ABC [Prop XVI. p 101.

Let O_2 A O_3 be the external bisector of the ∠ A,

O_1 B O_3 be the external bisector of the ∠ B,

O_1 C O_2 be the external bisector of the ∠ C,

and let these three lines intersect at O_1, O_2, O_3

Then O_1, O_2, O_3, are the ex-centres of the △ ABC [Prop XIX. p. 104.

Hence by Prop XVI. p 101 and Prop XIX p 104, the four ⊙s O, O_1, O_2, O_3, are the ⊙s required.

In connection with these four ⊙s of the △ ABC, the following properties are worth noting —

Let these ⊙s touch BC or BC produced at D, D_1, D_2, D_3.

Let a, b, c, be the sides of the △ ABC, opposite the ∠s A, B, C;

r, r_1, r_2, r_3, the radii of the ⊙s O, O_1, O_2, O_3,
and s the semi-perimeter of the △ ABC

[I]

(1) ∵ the whole perimeter is composed of two equal tangents each from A, B, C, to the ⊙ O,

∴ s=the sum of one tangent each from A, B, C,
=one tangent from A+the side BC
=one tangent from A+a

∴ the length of one tangent from A to the ⊙ O=$s-a$

Similarly the length of each tangent from B, C,
=$s-b$ $s-c$, respectively

(2) ∵ each of the two equal tans from A to the ⊙ O_1 is composed of AB+one tan from B, and AC+one tan from C respectively,

∴ the sum of the two tangents from A to the ⊙ O_1
=$AB+BD_1+AC+CD_1$
=the whole perimeter.

∴ each tangent from A to the ⊙ $O_1=s$

Likewise each tan from B, C, to the opposite escribed ⊙=s.

(3) ∵ s=each tangent from A to the ⊙ O_1
=$AB+BD_1$ or $AC+CD_1$

∴ $BD_1=s-c$ and $CD_1=s-b$

But it has been shewn in (1) that
$CD=s-c$, and $BD=s-b$

∴ $CD=BD_1=s-c$, and $BD=CD_1=s-b$

(4) ∵ the tangent from A to the ⊙ O=$s-a$,
and the tangent from A to the ⊙ $O_1=s$

∴ a=the difference of these tangents,
=the common tangent to these two ⊙s.

(5) $\because$ $\triangle$ ABC$+\triangle$ O_1BC$=\triangle$ AB$O_1+\triangle$ ACO_1

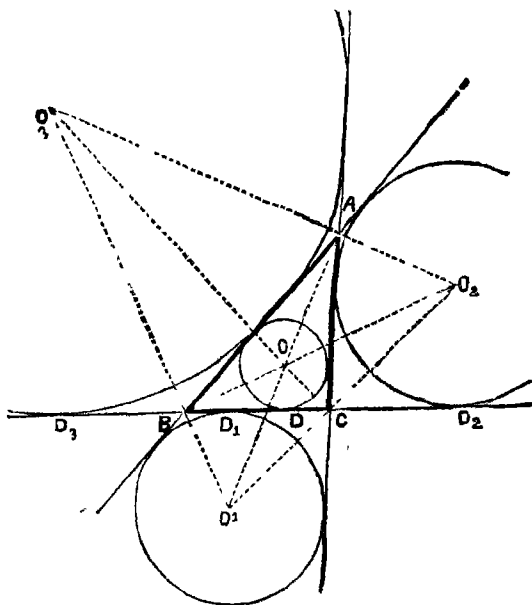
$\therefore$ $\triangle$ ABC$+\frac{1}{2}ar_1=\frac{1}{2}cr_1+\frac{1}{2}br_1$

$\therefore$ $\triangle$ ABC$=\frac{1}{2}(b+c-a)r_1$

$=(s-a)r_1$

Also, $\triangle$ ABC$=sr$ [*See Notes* p. 341.

$\therefore$ $\triangle$ ABC$=sr=(s-a)r_1$

Similarly $\triangle$ ABC$=(s-b)r_2=(s-c)r_3$

[II]

(1) It has been proved in Prop XIX, p. 104, that

A, O, O_1, are collinear,

B, O, O_2, are collinear,

C, O, O_3, are collinear.

(2) ∵ AOO_1 and O_2AO_3 are the internal and external bisectors of the ∠ at A,

∴ they are at right angles.

Similarly, $O_2B \perp O_1O_3$, and $O_3C \perp O_1O_2$.

∴ ABC is the pedal △ of △ $O_1O_2O_3$.

Hence all those properties proved in Props. *C* to *G* hold good with regard to the △s $O_1O_2O_3$ and ABC:—

(*a*) Of the four points O, O_1, O_2, O_3, each is the ortho-centre of the △ formed by the other three; and ABC is the pedal* △ of all these four △s.

(*b*) The rect. $OA.OO_1$ = the rect. $OB.OO_2$ = the rect. $OC.OO_3$.

(*c*) The ∠s subtended by O, O_3, at the side O_1O_2, are supplementary, etc.

(*d*) The ⊙ through O_1, O_2, O_3, = the ⊙ through O, O_1, O_2,
= the ⊙ through O, O_1, O_3,
= the ⊙ through O, O_2, O_3.

(*e*), (*f*), etc. etc.

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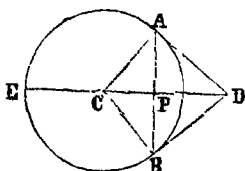
* Since the △ formed by joining the ex-centres O_1, O_2, O_3, of the △ ABC has for its pedal △ the original △ ABC, the △ $O_1O_2O_3$ may be called the antipedal triangle of ABC. This name was suggested to me for my little pamphlet on the evolution of "Pedal and Antipedal Triangles," by a friend Mr. Andrew Claude de la Chêrois Crommelin, F.R.A.S., an Astronomer of the Royal Observatory, Greenwich. It is obvious that $O_1O_2O_3$ has likewise its own antipedal △ and that also, and so on we get a series of antipedals upwards from the original △ ABC. Similarly ABC has its own pedal △ and that also has its own pedal, and so on we get a series of pedals downwards from the original △ ABC. I have mentioned in this pamphlet some remarkable properties of these series which seemed to be hitherto unknown—judging from the authority of some English Mathematicians and of Signor Luigi Cremona, Member of the Italian Parliament, and Prof. of Mathematics in the University of Rome—that the series of antipedals continually approaches in a particular manner an infinite equilateral △, and that the series of pedals continually diverges from an equilateral △ till it vanishes to a point (in certain cases after being turned into an oscillating straight line). Teachers, desiring to have a presentation copy of this pamphlet might apply to the Publishers of this treatise. Although the pamphlet makes use of Trigonometrical formulas, yet teachers might shew Geometrically some of these properties to their students.

A. S. G.

OBS Beginners may omit the remaining portion of this book.

Def *If any two points be taken on any diameter of a circle so that the rectangle contained by the distances of the centre from the two points be equal to the square on the radius, the points are called* ***inverse points*** *with respect to each other. The circle is called the* ***circle of inversion,*** *and the centre of the circle is called the* ***centre of inversion***

Proposition J Problem

To find the inverse of a fixed point with respect to a circle

Let C be the centre of the ⊙, and P a fixed point Join CP.

(1) Let P be inside the ⊙ Draw APB at right angles to CP

Join CA

Draw AD at right angles to CA, meeting CP produced at D.

The point D *is the inverse of* P

The rect CP, CD=the sq on AC [Prop II *Cor* p 186

∴ D is the inverse of P

(2) Let the fixed point be outside the ⊙, as D

Then draw AB the chord through the points of contact of the tangents from D , and let AB cut CD at P

Then the rect CD, CP=the sq on AC [Prop II *Cor* p 186.

∴ P is the inverse of D Q E F

OBS Hence, the inverse of a point outside a ⊙ is the point of intersection of its chord of contact and the diameter (produced) through that point

NOTE Since the ⊙ on CD as diameter cuts the given ⊙ at A, B, the chord of contact AB (for the external point D) is also the common chord of the two ⊙s We shall refer to this presently

Def *When the foot of the perpendicular drawn from a point at finite distance on a line is at an infinite distance from the point, the line on which the perpendicular is drawn is called the* ***line at infinity ;*** *and the foot of the perpendicular is a* ***point at infinity.***

[Compare this definition with that in p 285]

COR. The inverse of a point at the centre of inversion is a point at infinity.

For, the rect. CP, CD is constant, but when P is on C, CP is nothing, therefore CD is of infinite length. Hence D is a point at infinity.

[Compare this result with the point of intersection of the radical axis and the line of centres of two concentric circles, p. 254.]

Proposition K. Theorem

If two inverse points be taken on each of two or more lines drawn from the centre of inversion, the inverse points are concyclic.

Let C be the centre of inversion. E, A are inverse points on CA; F, B are inverse points on CB; and G, D are inverse points on CD; *the points* E, A, F, B, G, D *are concyclic.*

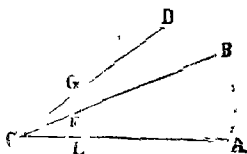
The square on the radius of the circle of inversion

=rect. CA, CE

=rect. CB, CF

=rect. CD, CG.

Therefore the points E, A, F, B, G, D are concyclic. [Prop. V. p. 349.

Wherefore, *if two inverse points* &c. Q. E. D.

Proposition L. Theorem

The inverse of a circle through the centre of inversion is a line; and conversely, the inverse of a line is a circle through the centre of inversion.

(1) Let ABC be a ⊙ through the centre of inversion C. Take any point B on the $\bigcirc^{ce}$ of the circle. Let CA be a diameter.

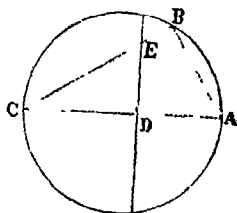
In CA take the point D the inverse of A, and in CB take E the inverse of B. Join ED. *The line* ED *is the inverse of the circle* ABC.

Because the square on the radius of the ⊙ of inversion = CE, CB =CA, CD,

∴ A, B, E, D, are concyclic. [Prop. V. p. 349.

Because ABC is a right angle, [III. 31.

∴ EDA is a right angle.

∴ the locus of E, the inverse of any point B on the $\bigcirc^{ce}$ ABC, is the line ED perpendicular to CA.

(2) Let ED be any line and let CD be perpendicular to ED. In CD produced take the point A the inverse of D, and in CE produced take B the inverse of E. *The inverse of* ED *is the circle* ABC *passing through* A, B, C.

The points E, D, A, B, are concyclic [Prop. *K*

∴ the sum of the ∠s EBA, EDA = two right angles [III 22.

But the ∠ EDA is a right angle

∴ the ∠EBA is a right angle

∴ the locus of B or the inverse of ED is a circle on CA as diameter

Cor The inverse of a locus is the locus of the inverses of all points on it

Def *The line passing through the inverse of any point, with respect to a circle, and cutting at right angles the diameter containing the point, is called the* **polar** *of the point with respect to the circle, and conversely, the inverse of the foot of the perpendicular from the centre of a circle on any line is called the* **pole** *of the line with respect to the circle*

Proposition M Theorem

If from a point outside a circle two tangents be drawn, the straight line joining the points of contact is the polar of the point from which the tangents are drawn

Let P be the point outside the ⊙ whose centre is C, PA, PB are tangents to the ⊙ Join AB AB *is the polar of* P

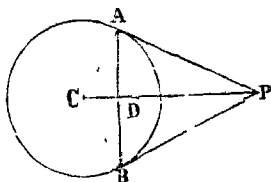
CP bisects AB at D at right angles

Because CAP is a right angle,

∴ CD.CP = the sq on AC

∴ the point D is the inverse of the point P [Prop. *J*.

∴ AB is the polar of P, and P is the pole of AB

Wherefore, *if from a point outside* &c Q. E D.

Cor 1 The perp to CP at P is the polar of the point D.

Cor 2. The polar of a point at the centre of inversion is a line at infinity

For, $CD.CP = CA^2$. Now, if P be taken on C, CP will be nothing, and CD will be of infinite length, hence D will be a point at infinity, and AD, perpendicular to CD, will be a line at infinity.

[Compare this with the radical axis of two concentric circles, p. 285.]

NOTE Comparing the Note in p 284 with the *Cors* to Props. *J* and *M*, the following coincidence is worthy of notice —

(1) The chord of contact is the polar of any pole external to the ⊙

Also, the chord of contact is the radical axis of the original ⊙, and of that drawn on the line joining its centre with the pole, as diameter

Hence when the pole is external to the original ⊙, *the radical axis of these two ⊙s is identical with the polar of that pole*

(2) When the pole lies on the ⊙ce its polar is the tangent at that point

Also, when the pole lies on the ⊙ce, the other ⊙ (namely that drawn on the line joining its centre with the pole, as diameter) touches the original ⊙ at that point

Hence the common tangent at that point is also the radical axis of these two ⊙s

Hence when the pole lies on the ⊙ce, *the radical axis of these two ⊙s is identical with the polar of that pole*

(3) When the pole is within the ⊙ce, the polar is external to the ⊙, being still at right angles to the line joining the centre with the pole Also, when the pole is within the ⊙ce the radical axis of the original ⊙ and of the other ⊙ (*i e* that drawn on the line joining its *centre* with *the pole, as diameter*) is *likewise* external to the original ⊙ and cuts the line of centres at the same point as the polar —

For, let O be the centre of the *original* ⊙, P the pole, and Q the inverse of P, (*i e* the point of intersection of the polar of P with OP), also let R=the radius

Let S be the middle point of OP, *i e* the centre of the ⊙ on OP as diameter

Then (*a*) considering the polar of P, we get

$$R^2 = \text{OP}\cdot\text{OQ}$$
$$= 2\text{OS}\cdot\text{OQ}$$

Again (*b*) considering the radical axis of the two circles, and supposing it to meet OP produced at Q_1, we have

$$R^2 - \text{OS}^2 = \text{OQ}_1^2 - \text{SQ}_1^2$$
$$\therefore R^2 = \text{OQ}_1^2 + \text{OS}^2 - \text{SQ}_1^2$$
$$= 2\text{OS}\cdot\text{OQ}_1 \qquad \text{[II. 7.}$$

But $R^2 = 2\text{OS}\cdot\text{OQ}$ [from (*a*)]

∴ Q and Q_1 are identical

∴ when the pole is within the ⊙ce, *the radical axis of these two ⊙s is identical with the polar of that pole*

Hence, in all cases wherever the pole might be, the polar is identical with the radical axis of the original ⊙ and of that drawn on the line joining its centre with the pole as diameter.

Also, note that when the pole is at the centre of inversion, the polar becomes a line at infinity, and in such a case the two ⊙s are concentric (the second becoming an indefinitely small ⊙ at the centre of the original one), hence their radical axis is also a line at infinity, p 285

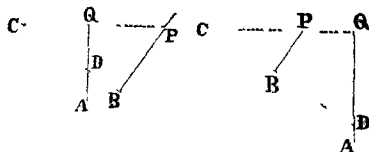
Obs ∵ the chord of contact of the tangents from the pole
=the polar
=the radical axis of the two ⊙s
=the common secant of the two ⊙s (through real or imaginary points of intersection of the ⊙s, p 284),

∴ it follows that although no real tangent can be drawn from a point within a ⊙, yet a real chord can be drawn (viz the polar of that point) through the imaginary points of contact of the tangents—although this chord will fall outside the ⊙

Proposition N Theorem

The pole of a line passing through a fixed point lies on the polar of that point, and, conversely, the polar of any point on a fixed line passes through the pole of that line

Let P be a fixed point and PB a line passing through P, C the centre of inversion, let QA be the polar of P

(1) *The pole of* PB *lies on* QA

Draw CB perp to BP, cutting QA at D
Because the ∠s DQP, DBP, are right angles,
∴ BDQP is cyclic [Prop. III p 348
∴ the point D is the inverse of the point B
∴ D is the pole of PB

(2) Let PB be a fixed straight line, C the centre of inversion
Take any point P in PB, and draw QA the polar of P
QA *passes through the pole of* PB
Draw CB perp to PB cutting QA at D
As before, the point D is the inverse of B.
∴ QA passes through the pole of PB
Wherefore, *the pole of a line* &c Q. E D.

Proposition O. Theorem

The line joining any two fixed points is the polar of the point of intersection of the polars of the points, and the point of intersection of any two lines is the pole of the line joining the poles of the lines

(1) Let A, B be two fixed points, C the centre of the circle of inversion; join CA, CB. Find D the inverse of A and E the inverse of B. [Prop. J.

Draw DF the polar of A, and EF the polar of B.

The pole of AB lies on DF, [Prop. N.

also the pole of AB lies on EF.

∴ F, the point of intersection, is the pole of AB.

∴ AB is the polar of F.

(2) Let F be the point of intersection of the two lines DF, EF; let C be the centre of the ⊙ of inversion.

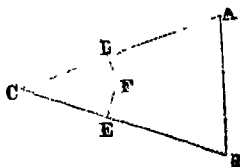
Draw CD perp. to DF, and CE perp. to EF.

Find A the inverse of D, and B the inverse of E.

A is the pole of DF, and B the pole of EF.

As before, AB is the polar of F.

∴ F is the pole of AB.

Wherefore, *the lines joining* &c. Q. E. D.

Proposition P. Theorem.

The square on the line joining the centres of the inscribed and circumscribed circles of a triangle, together with twice the rectangle contained by the radii, is equal to the square on the radius of the circumscribed circle.

Let ABC be a △.

Let O and Q be the centres of the circumscribed and inscribed ⊙s, respectively.

Join CQ and produce it to meet the O^ce of the circumscribing ⊙ at D.

Join DO and produce it to meet the ⊙ at E. Produce OQ both ways to meet the ⊙ at G and H. Join DB, EB. Draw QK perp. to BC.

QK is the radius of the inscribed ⊙.

Because the $\angle$ DQB = half of the $\angle$s ACB, ABC,
and the $\angle$ DBQ = the $\angle$s QBA, ABD
= the $\angle$s QBA, ACD
= half of the $\angle$s ABC, ACB

$\therefore$ the $\angle$ DQB = the $\angle$ DBQ, $\therefore$ DQ = DB.

The $\triangle$ DEB is equiangular to the $\triangle$ CQK

$\therefore$ the rect. DB, QC = the rect DE, QK [Prop VI. p 350.

$OH^2 - OQ^2$ = GQ QH [II 5.
= DQ QC [III 35.
= DB QC [for DQ = DB.
= DE QK
= twice the rect contained by the radii of the circumscribed and inscribed $\odot$s

Wherefore, *the square on the line &c* Q E D.

Proposition Q Theorem

The square on the line joining the centre of the circle circumscribed about a triangle with the centre of one of the escribed circles, is equal to the square on the radius of the circum-circle, together with twice the rectangle contained by the radii

Let ABC be a $\triangle$, let O be the circum-centre, and Q_1 an ex-centre, let Q be the in-centre Join OQ_1, cutting the circum-circle at F Let Q_1O produced meet the same $\odot$ at E

Produce CQ to meet the $\odot$ at D

Because each of CQ and CQ_1 bisects the $\angle$ ACB,

$\therefore$ CD and CQ coincide

Join OQ, BQ, and DB. Join DO, and produce it to H, join BH

Draw Q_1G perp to CB produced

Because BQ bisects the $\angle$ ABC, and BQ_1 bisects the $\angle$ ABG, $\therefore$ QBQ_1 is a right angle

As in the preceding Proposition, DQ = DB, and the $\angle$ DBQ = the $\angle$ DQB, $\therefore$ the $\angle$ DBQ_1 = the $\angle$ DQ_1B

$\therefore$ DQ = DB = DQ_1

The $\triangle$ CQ_1G is equiangular to the $\triangle$ DBH.

$\therefore$ $CQ_1 \cdot DB = Q_1G \cdot DH$ [Prop. VI. p 350.

$(Q_1O)^2 - (OF)^2 = EQ_1 \cdot FQ_1$ [II 6

$= DQ_1 \cdot CQ_1$ [III. 36. *Cor*.

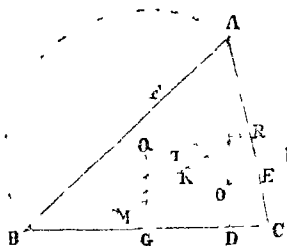
$= DB \cdot CQ_1$

$= Q_1G \cdot DH,$

= twice the rect contained by the radii.

Wherefore, *the square on the line* &c Q E D.

Proposition R Theorem

In any triangle, the point of concurrency of the medians of a triangle is collinear with the ortho-centre and the circum-centre, and its distance from the ortho-centre is double that from the circum-centre

Let ABC be a $\triangle$, AD, BE, CF perps on the sides, O the ortho-centre, Q the circum-centre R the middle point of AC, K the middle point of QO Let QG be perp to BC, M, middle point of BO Join BR, cutting QO at T

$\because$ QR is parallel to BE, for each of them is perp to AC,

$\therefore$ the $\triangle$ QTR is equiangular to the $\triangle$ BTO

$\therefore$ $TO \cdot QR = BO \cdot QT$. [Prop VI. p 350.

But BO is double of QR [Prop. *B*.

$\therefore$ TO is double of QT

Similarly BT is double of TR

$\therefore$ T is the point of concurrency of the medians [Prop. XVII. p 102.

Wherefore, *in any triangle*, &c. Q E. D.

Proposition S. Problem

To find a point on the circumference of a circle so that the sum of the distances of the point from two fixed points on the same circumference shall be a maximum

Let ABC be a ⊙, A, B, being two fixed points

Bisect the arc ACB at C C *is the required point*

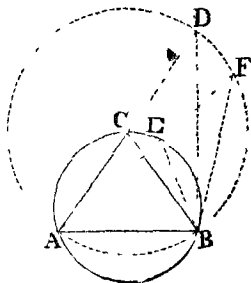
Join AC, CB

On AB describe the segment ADB containing an ∠ = half the ∠ contained in the segment ACB. Produce AC to meet ADB at D. Join DB. ∠ACB = ∠CDB + ∠CBD = 2∠CDB

∴ ∠CDB = ∠CBD

∴ CB = CD = AC

∴ AD is the diameter. Take any point E in the arc ACB, produce AE to F. EF = EB. AD is greater than AF

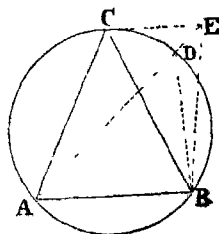
∴ AC + CB is greater than AE + EB. ∴ the point C the middle of the arc ACB is the required point. Q. E. F.

Proposition T. Problem

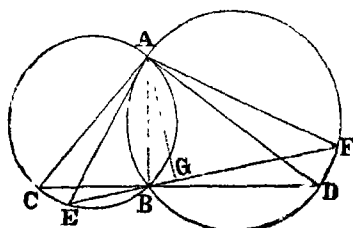
To find a point on the circumference of a circle so that the area of the triangle formed by joining the point with the ends of a fixed chord shall be the maximum

Let AB be a fixed chord of the ⊙ ABC. Bisect the arc ACB at C. ACB *is the required triangle*

In the arc CB take any point D. The tangent CE at C is parallel to AB. Produce AD to meet the tangent at E

△ ACB = △ AEB, which is greater than the △ ADB.

Wherefore C, the middle point of the arc ACB, is the required point.
Q. E. F.

Proposition U. Theorem

Of all triangles which can be formed by joining any point of intersection of two circles with the ends of the line passing through the other point of intersection and intercepted by the circles, the triangle formed by joining the ends of the longest line is the maximum.

Let the ⊙ ABC cut the ⊙ ABD at A and B let CD be at right angles to AB Draw any other line EBF terminated by the ⊙s Draw AG ⊥ EF. EF is less than CD (Prop 3, page 288), and AG less than AB ∴ the rect AB, CD is greater than the rect AG, EF ∴ △ ACD is greater than the △AEF Q E D

Proposition V. Theorem

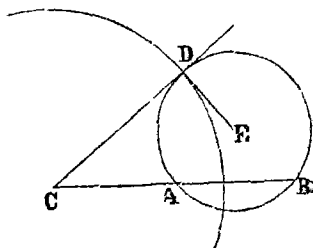
Every circle passing through a pair of inverse points with respect to a circle cuts that circle orthogonally

Let A, B be two inverse points with respect to the circle whose centre is C From C draw CD touching any ⊙ passing through A, B.

CA CB=CD^2=(radius of the ⊙ of inversion)^2

∴ CD=radius of the ⊙ of inversion

∴ the ⊙ passing through A, B, cuts the ⊙ of inversion orthogonally.

Q E, D.

MISCELLANEOUS EXAMPLES.

1 Of all triangles inscribed in a circle, the equilateral triangle has the maximum perimeter

2 Given two diagonals of a quadrilateral ; at what angle must they be placed so that the quadrilateral shall have the maximum area ?

3 Of all the three bisectors of the angles of a triangle the bisector of the greatest angle is the least.

4 Of all rectangles of a given area, the square has the minimum perimeter

5 Two fixed points are taken outside a given circle ; a variable point is taken on the circumference of the circle , the angle formed by joining the fixed points with the variable point is the maximum when the circle described about the three points touches the given circle externally, and the minimum when the circle about the three points touches the given circle internally

6 Given that the sum of the squares of two straight lines is equal to the square on a third , to find the two lines when the rectangle under them is the maximum.

7. Of all triangles inscribed in a given triangle, the pedal triangle has the minimum perimeter

8. Of all parallelograms having the same diagonals the rhombus has the maximum area

9 The intercept made on a variable tangent by two fixed tangents subtends a constant angle at the centre of the circle

10 Draw a tangent to a given circle so that the triangle formed by it with two fixed tangents shall have the maximum area.

11 Of all triangles that can be inscribed in a circle the equilateral triangle has the maximum area

12 Of all figures that can be inscribed in a given circle, the regular figure has the maximum perimeter.

13 Of all figures that can be inscribed in a given circle the regular figure has the maximum area.

14. Two circles cut each other ; to draw through one point of intersection a line the sum of the segments of which intercepted by the circles shall be the maximum.

Def *When the angles of a triangle are given, the triangle is said to be given* ***in species***

15 Given three fixed points, and the three angles of a triangle, to draw through the given points three straight lines which shall form a triangle given in species whose area shall be a maximum

16 In a given triangle to inscribe a triangle of given species whose area shall be a minimum

17 AB is a straight line outside the circle CDE, O is the centre of the circle, OB is drawn perpendicular to AB, AC and BE are tangents to the circle, shew that $AC^2 = AB^2 + BE^2$

18 If straight lines be drawn from a fixed point to all the points of the circumference of a given circle, the locus of all their points of bisection is a circle

19 If a circle touch the circumference of a semicircle at A and also the diameter of the same at B, and if BC be drawn at right angles to the diameter meeting the circumference at C, the square on BC is equal to twice the rectangle contained by the radii of the circles

20 The three circles, each of which passes through the ortho-centre and the extremities of each side of a triangle, are equal to one another and equal to the circum circle.

21 Describe a circle cutting two given circles orthogonally.

22 Describe a circle cutting three given circles orthogonally.

23 The common chords of any three intersecting circles are concurrent

24 If a straight line be drawn through the centres of two unequal circles cutting the circles in C, D, E, F respectively, then the rectangle contained by CE, DF is equal to the square on the direct common tangent

25 If two circles do not cut one another, any system of circles cutting them orthogonally always passes through two fixed points on the line joining the centres of the two circles

26 If from the centre of a circle, a straight line be drawn to any point in a chord, the square on that line together with the rectangle contained by the segments of the chord, shall be equal to the square on the radius

27. Given the base and the vertical angle of a triangle; to find the locus of the centre of the nine point circle.

28. Find the radical centre of three circles.

29 The sides of the triangle formed by joining the three ex centres, as well as the three lines joining the ex-centres with the in-centre, are bisected by the circum-circle

30 Given in position, the circum centre of a triangle, the centre of its nine point circle, and the middle point of the base; construct the triangle

31 Given in position, the ortho-centre of a triangle, the centre of the nine point circle, and the middle point of the base; construct the triangle.

32 Every circle passing through a pair of inverse points with respect to another circle is orthogonal to the other

33. If perpendiculars be drawn from any point on the circumference of a circle to two tangents and to the chord of contact, the square on the perpendicular to the chord is equal to the rectangle under the other two perpendiculars

34 Find the locus of a point such that if straight lines be drawn through it cutting a given circle, the rectangle under the intercepts between the point and the circle shall be constant

35 If the perpendiculars from any point on the circumference of the circle described about a triangle on the sides be produced to meet the circumference again, the straight line joining each point of meeting and the vertex of the angle opposite the side on which the perpendicular is drawn, is parallel to the Simpson's Line

36 Simpson's Line bisects the straight line joining the ortho-centre and the point on the circumference of the circum circle from which the perpendiculars are drawn

Def *The triangle the corners of which and the respective opposite sides are poles and polars, is called a* ***self-conjugate*** *triangle*

37 The ortho-centre of a self-conjugate triangle is the centre of inversion.

38 The three circles on the sides of a self-conjugate triangle as diameters cut the circle of inversion orthogonally

39 Through a fixed point inside or outside a circle a chord is drawn, and tangents are drawn at the ends of the chords; the locus of the intersection of the tangents is the polar of the fixed point.

40 When a triangle is such that two of its vertices and the sides opposite to them are poles and polars with respect to a circle, the third vertex and its opposite side are pole and polar to each other with respect to the same circle.

———o———

Miscellaneous Examples. Page 398.

Hints for Solution.

1. Let ABC be any triangle inscribed in the circle ABC.

That the perimeter of the triangle ABC be a maximum, the arc ACB should be bisected at C, the arc ABC be bisected at B and the arc BAC be bisected at A (Prop 8) Therefore $AC=CB=AB$

2 Let the two diagonals AC and BD cut each other at O Draw $EOG \perp BOD$ Make $OE=OA$, and $OG=OC$ The parallel to BD through E shall fall above AE &c

3 Let ABC be a triangle so that the angle $BAC >$ the angle ABC, the angle $ABC >$ the angle ACB Let the bisectors AD, BE, CF, intersect at O Draw $OG \perp BC$, $OH \perp AC$

$\angle OAF$ is greater than the $\angle ABO$, $\therefore BO > AO$

$\angle AEO = \angle C + \frac{1}{2}\angle B$, and $\angle ODG = \angle C + \frac{1}{2}\angle A$, and both are acute angles But $\angle ODG$ is greater than the $\angle OEH$, and $OG = OH$ $\therefore OE > OD$, $\therefore BE > AD$

4 Let AB be a side of the square, produce AB to C making $BC=AB$ Take any point D in BC $AD \cdot DC = BC^2 - BD^2$ $\therefore$ side of the square which is equal to the rectangle under AD, DC is less than the rectangle under AB BC, . . &c Also see Prop 5, p 290

5 Let A, B, be the fixed points, and F the variable point on the circumference By Ex 33, Book IV describe a circle passing through A, B and touching the given circle at F

6 On AB, the third line, describe a semicircle ADB $AD^2 + DB^2 = AB^2$ The rectangle $AD \cdot DB$ is the maximum when D is the middle point of the arc ADB Apply Prop V, page 189, and Prop 8

7 Suppose any triangle DEF be inscribed in the triangle ABC, so that D, E, F, be on BC, CA, AB respectively That the triangle EDF may have the minimum perimeter, the angle EFA should be equal to the angle DFB the angle FDB=the angle EDC, and the angle DEC=the angle AEF (Prop VII page 93) The bisectors of the angles EDF, DEF, EFD pass through a common point and also pass through the angles A, B, C (Props XVI and XIX pp. 101, 104) CF AD, BE are perpendiculars on the sides AB, BC, CA, respectively; $\therefore$ EDF is a pedal triangle; &c.

8 Apply Ex 2.

9 Let AB, AC be the two fixed tangents and PQ the intercept of a variable tangent. If O be the centre, the angle POQ is half of the angle BOC.

10 The tangent drawn at the middle point of the intercepted arc is the required one.

11 Proceed as in Ex 1, applying Prop T.

12 Proceed as in Ex 1

13 Proceed as in Ex. 11.

14. See Prop. 8, Page 288

15 Let A, B, C be the three fixed points, and D, E, F three angles of a triangle given in species On AB describe a segment ABG containing an angle$=D$, and on BC a segment BCK containing an angle$=E$ Complete the ⊙s and let them intersect at O The angle AOB is the supplement of D and BOC is the supplement of E The angle AOC is to be the supplement of F Describe a circle $AOCH$ about AOC the segment AHC contains an angle$=F$. Through B draw $GBC \perp OB$ Join GA, KC, and produce them to meet at H The angles GAO and OCK are right angles The point H lies on the arc of the segment on AC Apply Prop U

16 Let GHK be a triangle, D, E, F angles of a triangle given in species On GH describe a segment of a circle containing an angle$=\angle K+\angle D$, on GK describe a segment containing an angle$=\angle H+\angle E$ Let O be the point of intersection of the two segments The segment KOH of the circle about the $\triangle KOH$ shall contain an angle$=\angle G+\angle F$ Draw OA, OB, OC perpendiculars to GH, GK, KH The angle $GOH=\angle K+\angle OHK+\angle OGK$ $=\angle K+\angle BAC$

$\therefore$ $\angle BAC=\angle D$, likewise $\angle ABC=\angle E$, and $\angle ACB=\angle F$

$\therefore$ $\triangle ABC$ is given in species Because GHK is a maximum with respect to the $\triangle ABC$, the sum of the portions cut off by ABC is the maximum Hence ABC is the minimum triangle

17 Apply Euc. I 47

18. See Note to Prop IV, p 279

19 Apply Euc III 36

20 Let ABC be a triangle Let the perpendiculars AD, BE, CF meet the circum-circle at G, H, K respectively, let O be the ortho-centre. $OD=DG$ $\therefore$ the segment BOC of the circle described about OBC is equal to the segment BGC, &c

21. See Prop X, Book III p 286

22 Find the radical axis of any two of the circles and also the radical axis of one of them and the third. The point of intersection of the two axes is the centre of the required circle

23 Let AB, the common chord of circles ABR and ABD, cut CD the common chord of circles ABD, CDQ at O. Let ⊙ ABR cut the ⊙ CDQ at E and F EO produced will pass through F. If not, let EO produced cut the ⊙ ABR at H and the circle CDQ at G $AO \cdot OB=CO \cdot OD=EO \cdot OG$, $AO \cdot OB=EO \cdot OH$, $\therefore$ $OG=OH$, which is impossible

24 Let O be the centre of the larger circle and Q of the other, and C, D be in the larger circle and E, F in the smaller.

From CD cut off CM and DK, each $=QF$

Let AB be the direct common tangent. Draw radii OA, QB. Draw $QG \parallel AB$ $OG=OK=OM$.

$AB^2=OQ^2-OK^2=MQ.KQ=CE.DF$

25 Let A, B, be the centres of the circles; C a point on their radical axis CD which is $\perp AB$ Let CD cut AB at D Draw CE, CF tangents to circles whose centres are A and B respectively With C as centre and CE or CF as radius describe a circle cutting AD at G and DB at H $AC^2-CG^2=AD^2-DG^2=AC^2-EC^2=AE^2$. But AE is a fixed line and D a fixed point, $\therefore$ G is a fixed point

Obs. DG=tangent from D

26 Let AB be the chord of the circle whose centre is C D is any point in the chord AB, $CE \perp AB$ $CD^2+AD.DB=CE^2+DE^2+AD.DB=CE^2+AE^2=AC^2$

27 The angle subtended at the ortho-centre is supplementary to the vertical angle On BC the given base describe a segment BOC containing an angle=the supplement of the vertical angle, and also describe a segment BAC containing an angle=the vertical angle Let Q be the centre of the circum circle The middle point of the line joining Q with any point on the segment BOC is a centre of the nine point circle Find the locus of the centre (See Note to Prop IV, p 279)

28 See Note, page 285

29 See Fig to Prop. I p 384

Let ABC be the Δ, and O_1, O_2, O_3, the ex centres Then by the above Prop, ABC is the pedal triangle of the Δ $O_1 O_2 O_3$, and O is the ortho centre of $O_1 O_2 O_3$ Hence the circum-circle of ABC is the nine-point $\odot$ of $O_1 O_2 O_3$ (Prop II p 380), and therefore passes through the middle points of the sides of the Δ $O_1 O_2 O_3$, and also through the middle points of the lines joining O with O_1, O_2, O_3

30. Let Q be the circum centre, G the middle point of the base, K the centre of the nine point circle Draw BGC at right angles to QG Produce QK to O making $KO=QK$ O is the ortho-centre Draw $OD \perp BC$ Produce GK to meet DO at H Produce OH to A making $HA=HO$ With Q as centre and QA as radius describe a circle cutting BC at B and C ABC is the required triangle

31 Let O be the ortho-centre, K the centre of the nine-point circle and G the middle point of the side Produce OK to Q making $KQ=OK$ Join QG Draw BGC at right angles to QG, &c. as in the preceding exercise.

32 Apply Euclid III. 37.

33. Let AB, AC be any two tangents to the given circle BCD; D a point on the circumference. DE, DF, DG are perpendiculars to BC, AC, AB respectively $\angle DEF = \angle DCF = \angle CBD = \angle EGD$. $\angle EFD = \angle ECD = \angle DBG = \angle GED$ &c. Apply Addl. Prop. VI, page 350

34. Let C be the centre. Let P be a point,

(1) inside, (2) outside the circle. Let the line through P cut the circle at A, B

(1) $PC^2 + AP.PB = AC^2$ (Ex 26) $\therefore PC$ is constant.

(2) $PC^2 = AC^2 + AP.PB$ $\therefore PC$ is constant

$\therefore$ a ⊙ with centre C and radius CP is the required locus.

35 Let ABC be the circle circumscribing the triangle ABC, D a point in the arc AC, DE, DF, DG ⊥s to BC, AC, BA produced respectively. Produce DE to meet the circumference at H The quadrilateral $ECFD$ is cyclic $\angle EFC = \angle EDC = \angle HDC = \angle CAH$, $\therefore AH \parallel GFE$, &c.

36 Let D be the point in the arc AC of the ⊙ circumscribing the $\triangle ABC$, and DE, DF, DG be perpendiculars to BC, CA, AB respectively, let DG cut the ⊙ again at H Let Q be the circum-centre draw $QK \perp AB$ Let O be the ortho-centre Join GO, HC Produce GD to M making $DM = GH$ The perpendicular from Q on HD will bisect it, GM is double of QK But OC is double of QK

$\therefore OC = GM$

Let OC cut GFE at N By Ex. 35, $HC \parallel GE$ $\therefore GH = NC = DM$. $\therefore GD = ON$ $\therefore OD$ is bisected by GN

37 See Prop C $BO.OE = FO.OC = AO.OD$. The circle with centre O and radius equal to the line, the square on which is equal to any of the rectangles, is the circle of inversion.

38 The circle on any side as diameter passes through the feet of the perpendiculars on the other sides, $\therefore$ the ends of these perpendiculars and the ends of the side are inverse points with respect to the polar circle $\therefore$ the circle passing through the inverse points cuts the polar circle orthogonally (Prop. V)

39 Let AB, any chord of a ⊙ whose centre is C, pass through a fixed point P. T, the intersection of the tangents at A, B, and the middle point of AB, are inverse points with respect to the circle.

$\therefore$ the locus of the point is polar to P.

40. Let ABC be a triangle so that AC is a polar to B and AB is polar to C. Let BF and CE be perpendiculars to AC, AB. Let BE, CF produced meet at O. B, C, E, F are cyclic. $\therefore OB.OF = OC.OE$. OA is $\perp BC$. Let OA produced meet BC at D A, D, C, E are cyclic. $\therefore OC.OE = OD.OA$, &c.

THE THEORY

OF

MAXIMA AND MINIMA.

This difficult but interesting subject has received hitherto such a meagre and unscientific treatment that it has unfortunately become a stumbling block in the way of most students and has inspired them with a complete horror as it it were a regular *bête noir*. All that seems to have been done is the solution of a certain number of problems with a running commentary as to the peculiarities of some of them. This is certainly not enough, for it only *states* the answers to problems and shows them to be correct. Now students are as a rule well able to perform the mere verification of results if they are only given to them. What however they really want is the method of arriving unaided at the answers themselves. The method followed at present is something like this.—

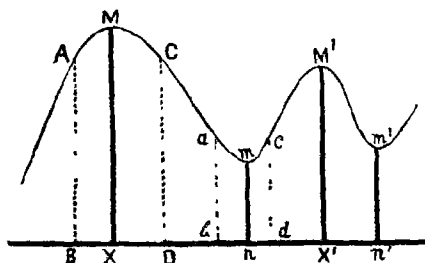
Supposing that the following problem had to be solved, "through a point within the arms of an angle, draw a straight line to cut the arms so that the rectangle contained by the segments of the intersecting line shall be a minimum" For the solution, the student is told to "draw a straight line through the given point so as to make equal angles with the arms", and then he is informed that "the segments on this line will contain the minimum rectangle, for, any other line through the point being taken, it is easily proved by Euc. III 35 that the rectangle contained by the segments of the latter is greater than the rectangle contained by the segments of the former. Therefore, the segments of the former line contains the minimum rectangle. Q. E. D."

The student naturally objects that it is easy enough for him to *prove* that the line making equal angles with the arms is the required line ; what he wants to know is what suggested the idea of drawing that particular line, for if left without any clue whatever he might have drawn any amount of lines

and constructed any number of geometrical figures—in fact gone floundering about hopelessly—till by accident he struck upon the right thing

To remedy the present evil we shall enter into such an analytical investigation into the Theory of Maxima and Minima as to discover if possible the laws governing them, an idea of which should afford the student almost an infallible solution to these questions

In page 92 we have given the definitions of Maximum and Minimum values of geometrical magnitudes We shall now examine these definitions more closely in order to obtain some clue to the laws we are seeking

Def. *When a geometrical magnitude, (a line, an angle or an area) subject to some given conditions, increases continuously for some time and then begins to decrease, it is a Maximum at the end of the increase also, if it decreases continuously for some time and then begins to increase, it is a Minimum at the end of the decrease*

Let us suppose that a man is climbing a range of hills AM*Camc*, let B*d* be the level of the ground

Then as he begins the ascent, his elevation above the level of the ground increases continuously till he arrives at the summit M, now if he continues his walk in the same direction he will begin to descend on the other side of the hill, and consequently his elevation will begin to decrease, till he reaches the bottom of the valley *m*; if he still goes

on walking in the same direction, he will ascend again, and therefore his elevation will begin to increase

Hence his elevation MX at the summit M is a maximum, since it ceases to increase at that point and begins to decrease (on the other side of the hill), moreover, his elevation mn at the bottom of the valley m is a Minimum, since it ceases to decrease at that point and begins to increase (on the other side of the valley)

Now the student must observe the following reasoning with close attention —

Just before he reaches the summit M his elevation is obviously somewhat less than at that point, that is, his elevation at A is somewhat less than his elevation at the summit M Similarly, after he has begun to descend on the other side of the hill, his elevation is also less than at the summit M Therefore, *there must be some point on the other side of the hill where his elevation is just equal to what it is at* A

In like manner, *just* before he comes to the bottom of the valley m his elevation is somewhat more than at that point, that is, his elevation at a is somewhat more than his elevation at m Similarly, after he has begun to ascend again on the other side of the valley, his elevation is also more than at the bottom of the valley m Therefore, *there must be some point on the other side of the valley where his elevation is just equal to what it is at* a

Let us then suppose that C is the point on the other side of M, so that

the elevation CD = the elevation AB,

and let c be the point on the other side of m, so that

the elevation cd = the elevation ab.

Now, if we choose the point A anywhere on the left side of M, it is plain that there is always a point somewhere on the right side of M, corresponding to A, so that the elevations at these points are equal Similarly, if we choose the point a anywhere on the left side of m, there is always a point somewhere on the right side of m, corresponding to a, so that the elevations at these points are equal.

[The corresponding point C need not be at the same distance from M as A is, that is, the equal values AB and CD *need not be symmetrical* about the Maximum value MX It will obviously not be so if the hill is rugged. Similarly, the corresponding point *c* need not be at the same distance from *m* as *a* is, that is, the equal values *ab* and *cd* need not be *symmetrical* about the Minimum value *mn*]

Hence we arrive at the following most important law :—

[I] On each side of a position of Maximum (or Minimum) value there are positions of equality—not necessarily at equal distances from the Maximum (or Minimum) value, but **somewhere** in the neighbourhood

Again, since to every point on the left side of M, there is a corresponding point on the right side of M, it follows that even the point *adjacent* to M on the left side has its corresponding point on the right side—and also adjacent to M, since we know from our definition of a Maximum value that the change in the magnitude must be *continuous* and *not disjointed* Similarly, the point adjacent to *m* on the left side has its corresponding point adjacent to *m* on the right side

But it has been already stated that adjacent or consecutive points on a line may be considered to be coincident

(See *Notes*, Book III p 270)

Hence we deduce this very important law —

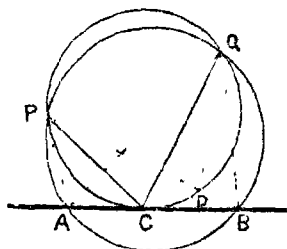
[II] The position of Maximum (or Minimum) value is nothing else than the position of coincidence of the two equal values mentioned in [I]

The right application of these two laws will almost invariably enable the student to solve a certain class of problems on this subject—provided of course that he knows the fundamental propositions of Geometry

We shall work out a few cases in order to illustrate the method of procedure —

Class A.

Proposition 1. Problem

To find the point in a given straight line, at which the lines joining it with two fixed points on the same side of the given line shall make a maximum angle.*

Let AB be the given straight line, and P, Q, the fixed points on the same side of AB

It is required to find the point in AB at which the lines joining it with P and Q shall make a maximum angle

Analysis.

Let C be the required point, so that

the ∠ PCQ is a maximum

We have to find the position of the point C.

Now, by our law [I],

since the ∠ PCQ is a maximum,

∴ on each side of the ∠ PCQ there are positions of equality,

that is, there must be two such points A, B, on each side of C, so that

the ∠ PAQ = the ∠ PBQ.

Hence, we infer from our knowledge of Geometry that a ⊙ must pass through the points P, A, B, Q.

[Addl Prop. I, p. 347.

∴ the line AB is a secant to this ⊙.

Again, by what has been said in our investigation above, if we had chosen the point A adjacent to C on the left side, its corresponding point B must also have been adjacent to C on the right side

In other words, by our law [**II**]

the point C is nothing else than the point where the pairs of equal angles like PAQ, PBQ, come and coincide

In consequence, AB which was a secant to the ⊙ PABQ, *becomes the tangent to the* ⊙ PCQ,

[*See Notes* Book III,* p 270

that is to say, a ⊙ passing through P, Q, touches the given line AB at the required point C

Hence we obtain the clue for the following construction —

Synthesis.

Describe a ⊙ passing through the fixed points P Q, and touching the given line AB

Let it touch at C

Then the ∠ PCQ is a maximum

For, taking any other point B in the given line AB we can shew that the ∠ PCQ is greater than the ∠ PBQ †

Q E F

*[Notes, Laws, etc marked with asterisks thus **, should be omitted by beginners]*

**NOTE Let us now return to the case of the man climbing the range of hills We have seen him pass over the bottom of the valley at *m*, and begin to ascend again on the other side If he goes on towards the same direction, he will come to a point M′ where he

---

† It is not necessary to go through the whole of the synthesis In fact, once the theory is well understood, it is quite sufficient to give the mere construction of the synthesis

However to shew that the theory is correct, we shall finish the synthesis

Let PB cut the ⊙ PCQ at D, join QD

Then the ∠ PCQ = the ∠ PDQ [III 21

But the ∠ PDQ is greater than the ∠ PBQ [I 16.

∴ the ∠ PCQ is greater than the ∠ PBQ

∴ the ∠ PCQ is a maximum

will cease to ascend and begin to descend once more Hence, according to our definition, M′ is also a position of Maximum Similarly, if he still goes on, he will come to another position of Minimum at m'

Hence it would appear that in certain cases it is possible to have more than one value fulfilling the definition of Maximum or Minimum Of course the student might object that since M′ X′ is not so great as MX, the position at M′ should not be considered to be a Maximum at all However this objection is not valid *if we adhere to our definition* for, by Maximum is not meant the *greatest value under all circumstances*, but merely *greater than what it is immediately before and after that position*, and similarly for a Minimum In other words we have a Maximum value whenever there is *a change from an increase to a decrease* and Minimum, whenever there is *a change from a decrease to an increase* However in most cases coming under the first Four Books of Euclid (with some exceptions), there is only one Maximum value and only one Minimum value which we may then call *the* Maximum or *the* Minimum, that is, *the greatest value possible* or, *the least value possible*, respectively

If we examine the diagram carefully as to the configuration of the points M, m, M′, m' we shall see that ***in those cases in which there is more than one Maximum or Minimum, the Maximum and Minimum positions occur alternately*** In other words, ***two Maximum values cannot occur in succession, but must have a Minimum value between Similarly, two Minimum values cannot occur in succession, but must have a Maximum value between*** [**XXI**]

[The student may learn hereafter that the above is well illustrated in the physical fact that "positions of stable and unstable equilibrium occur alternately" – the former corresponding to a Minimum, and the latter to a Maximum The student may take a book and lay it vertically on a table so that it may be supported by one of its edges, then let him roll the book along the table but always in the same vertical plane Then the book will be supported in succession by an edge, a corner, an edge, a corner, etc The edges will give stable equilibrium, and the corners unstable Moreover, the height above the table of the centre of gravity of the book will be in positions of Minimum for the edges, and of Maximum for the corners]

We shall now resume our last Problem, to see if what we have just said apply in this case

The required point C was the point where the ⊙ PCQ, described to pass through P, Q, touched the given line AB

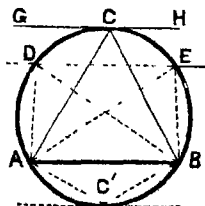
But another ⊙ may be described to pass through P, Q, so as to touch the line AB Let this ⊙ touch AB at the point C′, then the ∠ PC′Q is also *a* Maximum

[Of course this point C′ will be very much to the left of A.]

Again, supposing that we have to find such a point in AB as will subtend a *Minimum* angle at P, Q it is obvious that the required point is where the line joining Q, P cuts AB, for the angle made by the lines joining it to P and Q is nothing, hence in this case this point gives not only *a* Minimum angle, but also *the* (only) Minimum angle

It is obvious from what has been said in [**III**] that this point must lie somewhere between the Maximum positions C and C'; since two Maximum values must have a Minimum value between. The student might satisfy himself of this fact and find the position of C' by describing another ⊙ to pass through P, Q, so as to touch the line AB Then the point of intersection of the line QP (produced), with the line BA, lies between C and C'.

Proposition 2. Theorem

Of all triangles on the same base and with equal vertical angles, the isosceles is a maximum

On the given base AB describe a segment ACB containing the vertical angle

Then the vertices of all the △s will lie along the arc ACB

Let the △ACB be a maximum

We have to shew that the △ ACB must be isosceles.

Because ACB is a maximum,

∴ on each side of it there are positions of equality, by our law [**I**]

Let ADB and AEB be two such equal △s.

∴ the secant DE is parallel to AB. [I. 39.

Now, by our law [**II**], the point C is no other than the point where the vertices of a pair of equal △s (such as ADB, AEB) come and coincide

Hence, at C the secant passing through the vertices of a pair of equal $\triangle$s, *and which is always parallel to the base* AB, becomes a tangent.

Hence, we have this result—*the tangent at the vertex of the maximum triangle is parallel to the base.*

∴	the $\angle$GCA = the $\angle$CAB ,	[I 29.
also,	the $\angle$GCA = the $\angle$CBA	[III 32.
∴	the $\angle$CAB = the $\angle$CBA,	
∴	CA = CB	[I 6

Hence the maximum $\triangle$ ACB is isosceles. Q. E. D

NOTE Since the vertices of all these $\triangle$s lie along the arc ACB, the vertex which lies on A (or B) gives the *minimum* $\triangle$, for then the $\triangle$ has no area. Hence in this case, we have *one* maximum $\triangle$, and *two* minimum $\triangle$s—*the maximum lying between the two minimum*, in accordance with [III**]

Proposition 3. Theorem.

Of all triangles inscribed in a circle on a fixed chord as base, the maximum is isosceles.

(See the figure of the last Prop)

Let AB be the fixed chord of the ⊙ ACB.

Suppose ACB to be the maximum $\triangle$.

Now the proof that ACB must be isosceles, is identically the same as in the last Prop

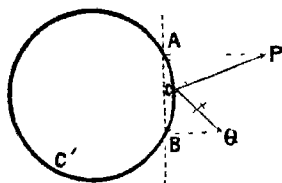
**NOTE Since the chord AB divides the ⊙ into *two* segments, there should be *two* such maximum $\triangle$s, one in each segment Hence the isosceles $\triangle$ AC′B in the other segment is also a maximum—the tangent at C′ being likewise parallel to AB

And as we have shewn in the Note to the last Prop , when the vertex of the $\triangle$ lies on A and B respectively, we have *two minimum* $\triangle$s

Hence in this case we have two maximum and two minimum $\triangle$s altogether , and of the former, both are isosceles, but one lying in either segment

Also, these four values occur ***alternately***, in accordance with [**III**] Namely, if we begin with any vertex, for instance C (maximum), and see along the arc in either direction, for instance from left to right as the hands of a watch move—we shall find in succession the $\triangle$s having the following vertices .—B (minimum), C′ (maximum), A (minimum), C (maximum), etc.

Proposition 4. Problem

To find a point on the circumference of a circle so that the lines joining it with two fixed external points shall make a maximum angle

Let P, Q, be the fixed points external to the ⊙ ACB

We have to find such a point on the ○ce as the lines joining it with P, Q, will make a maximum angle

Suppose C to be such a point,

so that the ∠ PCQ is a maximum

We have to find the position of C

Then by our law [I] on either side of C there are points on the ○ce which will give equal angles

Let the ∠s PAQ, PBQ, be these equal angles

Hence, a ⊙ passes through the points P, A B, Q

Hence the line AB is a common secant to this ⊙ and the given ⊙

Now, by our law [II], the point C is no other than the point where a pair of such equal angles coincide

Hence at the point C, AB becomes a common tangent to the ⊙ PCQ and the given ⊙

∴ the ⊙ PCQ touches the given ⊙ at C,

in other words, the point C is nothing else than the point where a ⊙ passing through the fixed points P Q, touches the given ⊙

This gives us the position of C

OBS Now, an *apparent* difficulty arises Our two laws [I] and [II] apply *both* for maximum and minimum values Hence, after we have come to a solution we have to ask ourselves the question, "is this result a maximum or a minimum," since the same method has to be followed for both For instance in the above case, had we desired to find the *minimum* angle, we would have proceeded in exactly the same manner, thus —

"Let the ∠ PCQ be a minimum
We have to find the position of C
Then by our law [**I**],
the two ∠s PAQ, PBQ, on either side, are equal
Hence AB is the common secant of the ⊙ PABQ and the given ⊙

Hence by our law [**II**] AB becomes a common tangent at C to the ⊙ PCQ and the given ⊙

∴ C is the point of contact of the ⊙ passing through P, Q, with the given ⊙"

Hence it would *appear* that the point of contact of the ⊙ passing through P, Q, and touching the given ⊙, gives the maximum angle *as well as* the minimum angle But this difficulty is easily explained For *two* ⊙s may be drawn, passing through P, Q, and touching the given ⊙—in one case the described ⊙ and the given ⊙ will be external to each other, and in the other case the given ⊙ will be wholly within the described ⊙

Let the two points of contact be C and C′ (that is let A and B come and coincide at C in the first case, and at C′ in the second case)

Hence of the two ∠s PCQ, PC′Q one is obviously a maximum and the other a minimum As to which one is which, it easily follows, since the ∠ PCQ is greater than the ∠ PC′Q, that the ∠ PCQ is a maximum, and the ∠ PC′Q a minimum

NOTE The objection might be raised that *both* these angles may be *maximum* or *both minimum*—there being nothing in the methods of [I**] and [**II**] to exclude this supposition However we have already said that two maximum values cannot occur successively without a minimum value intervening between, and similarly two minimum values cannot occur successively without a maximum value coming between But in this case no such intervention takes place (since it would require at least three distinct solutions), hence this objection cannot stand

**EXERCISES

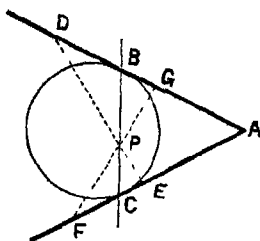
1 If the line joining P, Q cut the given circle, find the points on the circumference which give the maximum and minimum angles, and shew that there will be *two* points for maximum, and *two* for minimum these four points occurring alternately

2 Hence deduce the case when the line joining PQ is a tangent to the given circle

3 If the points P, Q, lie within the given ⊙, find the points on the circumference which give the maximum and minimum angles, and shew that there will be *two* points for maximum, and *two* for minimum, these four points occurring alternately

4 In the last case (when the fixed points P, Q, are within the given ⊙) is it ever possible that there should be only one maximum angle? In other words, is it ever possible that only one ⊙ can be drawn to pass through two internal points and touch the given ⊙?

Proposition 5. Problem.

Through a point within the arms of an angle, draw a straight line to cut the arms so that the rectangle contained by the segments of the intersecting line shall be a minimum.

Let AB, AC, be the arms of the given angle, and let P be any point within them

Suppose BPC to be the required line,
so that the rect BP PC is a minimum

We have to find the position of BPC

Now by our law [I],

on either side of BPC we can draw lines, the rectangle contained by the segments of which are equal.

Let DPE, GPF, be two such lines,
so that the rect DP PE = the rect GP PF

Hence a ⊙ passes through the points D, G, E, F
[Prop IV p 348.

∴ AB, AC, are secants to this ⊙.

Now by our law [II],

a pair of such lines as DPE, GPF, come together and coincide, and thus form the line BPC.

Hence AB, AC, become tangents to the ⊙ whose chord of contact is BPC.

But the two tangents AB, AC, are equal,
∴ the △ ABC is isosceles,
∴ the ∠ ABC = the ∠ ACB.

Hence the required line BPC is obtained by drawing though the given point P a straight line which makes equal angles with the arms AB, AC.

NOTE To shew that the rect BP PC is a *minimum*, and not a *maximum*, we have only to shew that it is *less* than the rectangle contained by the segments of any other line in its immediate neighbourhood For instance, the rect GP PF is obviously greater than the rect BP PC (III 35).

EXERCISES

1. Draw the required straight line when the given point P is *outside* the arms of the angle BAC

**2 In this Proposition (when the point P is within the arms) shew that if the line through P cut the arms or the arms produced, then there will be *two* cases of minimum, *and also two cases of maximum* these four occurring *alternately*

**3 Hence deduce the case when P is not within the arms, but the segments are to be made with the arms, or arms produced

Proposition 6 Problem.

In the arc of a segment of a circle to find the point from which the perpendicular on the chord is a maximum

(*See the first figure of Prop 25, Book III*)

Let ABC be the arc of the segment, and AC the chord

We have to find the point on the arc ABC from which the perpendicular on the chord AC is a Maximum

Suppose BD to be the required perpendicular

We have to find the position of the point B.

If on the arc AB we take a point E, and on the arc BC a point F, so that the perpendiculars from E, F, on AC are equal (by our law **I**)

then the secant through E, F, is parallel to AC

Hence when these equal perpendiculars come and coincide at the point B (by our law **II**),

the secant—which always remains parallel to AC—becomes the tangent at B

∴ the tangent at the required point B is parallel to the chord AC.

Hence it easily follows that the required point B is the middle point of the arc ACB Q. E. F.

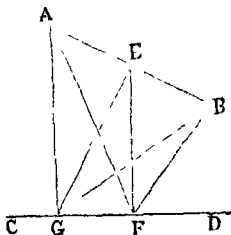
EXERCISE

Find the points on the circumference of a circle from which perpendiculars on a fixed chord are Maximum and Minimum.

Class B.

There is another class of problems in which we have to find the Maximum or Minimum values of *two quantities which vary together*, that is, as one of them changes in value, the other also changes—not necessarily in the same manner or degree, but only *simultaneously* The following is such a problem, and the student should carefully observe its method of proof, for we shall presently deduce a rule for the working of problems of this class

Proposition 7 Problem

To find a point in a given straight line so that the sum of the squares on the lines joining the point with two fixed points may be a minimum

Let CD be the given straight line,
and A, B, the fixed points

We have to find such a point in CD that the sum of the squares on the lines joining it with A, B, shall be a Minimum

Now we know that if *any* point G be taken in CD, and AB be bisected at E,

then in the $\triangle$ AGB,

$AG^2+BG^2=2AE^2+2EG^2$. [Prop III p. 187.

But $2AE^2$ is always constant, since AE is constant (AB being fixed).

$\therefore$ $AG^2+BG^2=$ a constant $+2EG^2$

$\therefore$ AG^2+BG^2 is a minimum

when $2EG^2$ is a minimum,

that is, when EG^2 is a minimum,

[for 2 being a constant quantity, its omission cannot affect the question]

that is, when EG is a minimum.

In other words, $AG^2 + BG^2$ is a minimum, when the line EG, drawn from E to the straight line CD, is a minimum, that is, when EG is perpendicular to CD

[Prop. VI p 92

Hence the required point is the foot of the perpendicular from E, the middle point of the line joining AB, on the given line CD

Now it may be noted that in the above case, there are *two quantities* (AG^2, BG^2,) *which change at the same time*, that is, we may express this briefly by saying that in the above case there are *two variables*

In the proof, we have shewn that *in all cases*

$$AG^2 + BG^2 = \text{a constant} + 2EG^2 ,$$

and then we have found under what condition $2EG^2$ is a minimum

Hence what we have done is merely to shew that *the expression involving the two variables* is always equal to *an expression involving only one variable quantity together with a constant* Now, after we have reduced the two variables into only one variable (together with a constant), it is easy enough to find under what condition that single variable gives a maximum or minimum result, according to the question

Hence, **in all problems of this class, the method of procedure is to reduce the expression involving the two variables into an expression involving a single variable, and then to find under what condition this single variable gives a maximum or minimum result.**

EXERCISES

1 Find the point on the circumference of a circle so that the sum of the squares on the lines joining it with two fixed external points, may be a Minimum, find also the point which gives a Maximum

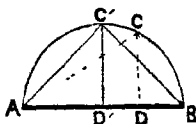
2 Hence deduce the case when the fixed points are within the circle, or lie on the circumference.

3. Where must the two fixed points be situated so that the sum of the squares on the lines joining it with a variable point on the circumference may undergo no change, but remain constant? What is this constant value?

4. If A, B, be two fixed points on a plane, and P *any* point on the same plane, find the least possible value of AP^2+BP^2.

5. Hence divide a straight line so that the sum of the squares on the two segments shall be a Minimum.

Proposition 8. Problem.

The sum of the squares on two lines is constant; to find the condition that the rectangle contained by the lines shall be a maximum.

Let AC, CB, be the given lines.

Then $AC^2+CB^2=$ constant $=AB^2$ (suppose).

Hence, if a ⊙ be described on AB as diameter, C must always lie on the ⊙ (since $AC^2+CB^2=AB^2$, for ACB is a right angle).

We have to find under what condition the rect. AC, CB shall be a maximum.

The rect. AC, CB = the rect. AB, CD †

But AB is constant.

Hence, the rect. AC, CB = a maximum

when CD = a maximum,

that is, when C is the middle point of the arc, as C′.

[Prop 6 p 417.

In that case, AC′ = C′B.

Hence, when the sum of the squares on two lines is constant, the rectangle contained by the lines is a maximum when the lines are equal.

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† For, since the ∠ ACB = a right angle,

∴ the rect. AC, CB = double the △ ACB

= the rect. AB, CD.

The above Prop may be expressed algebraically, thus —

If a^2+b^2=constant,
then ab=a maximum
when $a=b$

The student might have noticed that we have given similar algebraical propositions in pp 189 290, but have refrained from giving *proofs* of them, for the obvious reason that it would have been too premature before we had established our present theory

However we proceed to prove them all now.

(1) To shew that if $a+b$=constant,
then ab=a maximum
when $a=b$.

[Algebraical Illustration of Prop V p 189]

When ab is a maximum, $4ab$ is also a maximum since 4 is a *constant* quantity and therefore cannot affect any mutual relationship between the *variable* quantities a, b.

But $4ab=(a+b)^2-(a-b)^2$

∴ $4ab$=a maximum

when $(a+b)^2-(a-b)^2$=a maximum

But $(a+b)^2$ is constant, since $a+b$ is constant.

Hence $(a+b)^2-(a-b)^2$=a maximum
when $(a-b)^2$=a *minimum*.

[For it is obvious that the smaller $(a-b)^2$ becomes, the more will remain after *subtracting* it from the constant quantity $(a+b)^2$]

And $(a-b)^2$=a minimum
when it is equal to zero,
that is, when $a=b$
Hence, if $a+b$=constant,
then, when $a=b$,
$(a-b)^2$=a minimum,
that is, $(a+b)^2-(a-b)^2$=a maximum,
that is, $4ab$=a maximum,
that is, ab=a maximum.

(2) To shew that if $a+b$=constant,
then a^2+b^2=a minimum
when $a=b$

[Algebraical Illustration of Prop VI p. 189]

Now, if $a+b$=constant
then $(a+b)^2$=a constant.
$\therefore$ a^2+b^2+2ab=a constant
$\therefore$ a^2+b^2=a constant$-2ab$.
$\therefore$ the left side of this equation=a minimum,
when the right side becomes a minimum,
that is, when $2ab$ becomes a *maximum*

[For the larger $2ab$ becomes, the less will remain after *subtracting* it from a constant quantity]

Hence a^2+b^2=a minimum
when $2ab$=a maximum
that is, when ab=a maximum
that is, when $a=b$ [From (1)]

(3) To shew that if ab=constant,
then $a+b$=a minimum
when $a=b$

[Algebraical Illustration of Prop 5. p 290]

Now, if ab is constant, $4ab$ is also constant.
But $4ab=(a+b)^2-(a-b)^2$.
$\therefore$ $(a+b)^2-(a-b)^2$=a constant.
$\therefore$ $(a+b)^2$=a constant$+(a-b)^2$.
$\therefore$ the left side of this equation is a minimum,
when the right side becomes a minimum,
that is, when $(a-b)^2$ becomes a minimum,
that is, when $a=b$.
Hence, $(a+b)^2$=a minimum,
when $a=b$.
$\therefore$ $a+b$=a minimum
when $a=b$.

NOTE This is the converse of (1) Hence we may infer these important results —

From (1) we see that if the perimeter of a rectangle be constant (for the perimeter is equal to twice the sum of length and breadth), the area is a maximum when length=breadth, that is, when the rectangle becomes a square

Again, from (3) we see that if the area of a rectangle be constant its perimeter is a minimum when length=breadth, that is, when the rectangle becomes a square

(4) To shew that if a^2+b^2=constant,

then ab=a maximum

when $a=b$

Now if ab become a maximum, $2ab$ will also become a maximum

But $2ab=a^2+b^2-(a-b)^2$,

$\therefore$ $2ab$=a constant$-(a-b)^2$

Hence, the left side of this equation is a maximum,

when the right side becomes a maximum,

that is, when $(a-b)^2$ becomes a *minimum*

[For, the smaller $(a-b)^2$ becomes, the more will remain after *subtracting* it from a constant]

Hence, $2ab$=a maximum

when $(a-b)^2$ becomes a minimum,

that is when $a=b$.

$\therefore$ if a^2+b^2=*constant*,

then ab=a maximum

when $a=b$.

Def. *Two points are the* **images** *of one another with respect to a straight line, when the line joining the two points is bisected at right angles by that straight line.*

For instance, in Prop VII. p 93, E is the image of D on the line AB; and D is the image of E on the line AB.

NOTE. This definition is taken from a physical fact. For if AB were a mirror (facing D), then the image of an object at D would appear at E, and if the mirror AB were facing E, then the image of an object at E would appear at D.]

Def. *If there be two fixed points on the same side of a given line, then the point on that line, the straight lines joining which with the fixed points make equal angles with the given line, is called the* **point of reflexion** *of the two fixed points, with regard to each other, on the given line.*

For instance, in Prop VII p. 93, P is the point of reflexion on the line AB of the points C, D, with regard to each other

NOTE This name is given from the physical fact that P is the point in AB where a ray of light coming from C must be reflected in order to pass through D, and is also the point where a ray coming from D must be reflected in order to pass through C This arises from the law of reflexion of light that the initial and reflected rays make equal angles with the perpendicular to the reflecting surface, and therefore with the reflecting surface itself For instance, CP, PD, must make equal angles with AB, in order that P may be the point of reflexion of C, D, on AB

It is obvious that the point of reflexion P is nothing else than the point of intersection with AB of the line joining any one of the points C to the image E of the other point D

The geometrical truth established in Prop VII p 93, will enable us to deduce an important property of rays of light, and the physical fact will in turn suggest to us some geometrical results

Since of all points in AB, P is such that PC+PD is the least possible, it follows that a ray of light in passing from one point to another (after reflexion) seeks *the shortest path possible* *

We shall make use of the above physical phenomenon to deduce the following proposition from Prop VII p. 93

Proposition 9. Problem.

Given two fixed points without a circle, to find the point on the circumference so that the sum of its distances from the fixed points may be a minimum

The required point is obviously the point of reflexion. And the point of reflexion is that where the lines joining it with the fixed points make equal angles with the tangent.

Q. E F

* The student may learn hereafter that this is only a particular instance of a law of nature which states that motions always take place in the way of least resistance And a time may yet come when this again will be shewn to be merely an infinitesimal corrollary of the one grand universal law of the created world, which for the want of a better name we shall call the law of "Conservation of Energy"—understanding Energy not merely in the material sense, but also in the moral order of things.

Sometimes problems arise in which we have to find a Maximum or Minimum value of a geometrical quantity which *depends upon* two variables, but is not exactly an expression *involving two variables simultaneously*. In such a case the easiest plan is to make the geometrical quantity depend upon only one variable by means of successive reductions. This kind of simplification is analogous to our OBS. p. 108.

We shall illustrate the above method by the following example :—

Proposition 10. Problem.

In a circle there is a fixed chord, and through one end of it a variable chord passes (so that the angle between the two chords always changes). A parallelogram is described of which these two chords are adjacent sides. Find the greatest possible length of that diagonal which passes through the point of intersection of the two chords.

(See the figure of Prop. IV. *page* 279.)

Let AB be the fixed chord, and let a variable chord pass through the point B.

We have to find the maximum diagonal through B of the parallelogram whose adjacent sides are these two chords.

Let BD be any position of the variable chord, so that AD is the other diagonal. Let AD be bisected at G.

Now, since the two diagonals of a parallelogram bisect each other,

∴ the diagonal through B = double the line joining B, G.

Hence the question reduces itself to the following—"In a circle there are two chords AB, AD, of which AB is fixed but AD is variable. To find (double) the longest line joining B with the middle point of AD."

But we know that the locus of the middle point of AD is a circle [⊙ AGF in Prop. IV. p. 279.]

Hence the question is reduced still further into finding (double) the longest straight line that may be drawn to the ⊙ AGF from the external point B.

The longest line that may be drawn to this ⊙ from B, is that which passes through the centre H. [III. 8.

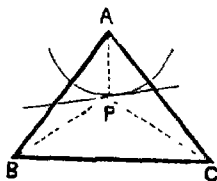
Hence, double this line is the longest possible diagonal through B of the series of parallelograms. Q. E. F.

Class C.

There is a third class of problems which is worthy of notice The characteristic feature of this species of problems is that a Maximum or Minimum result is to be obtained where three quantities are changing simultaneously, in other words, where *three variables are involved.*

We shall lay down the rules for the working of such problems from the following investigation —

Proposition 11 Problem

To find the position of a point such that the sum of its distances from the vertices of a triangle may be a minimum.

Let ABC be the $\triangle$.

We have to find the position of a point P such that

PA + PB + PC shall be a minimum

First, let us suppose PA to remain constant (in length) but PB, PC, to vary—that is, P to lie on the $\bigcirc^{ce}$ of a circle whose centre is A

In such a case PB + PC is a minimum when PB, PC, make equal angles with the tangent at P. [Prop. 9, p. 424.

And since AP is at right angles to the tangent at P,

$\therefore$ the $\angle$ APB = the $\angle$ APC

Hence, when PA remains constant (in length) but PB, PC, vary,

then PB + PC is a minimum

when PB, PC, make equal angles with PA.

Similarly, when PA, PC, vary (but PB remains constant),

then PA + PC is a minimum

when PA, PC, make equal angles with PB.

Likewise, when PA, PB, *vary* (*but PC remains constant*),

then PA + PB is a minimum

when PA, PB, make equal angles with PC

∴ *When* PA, PB, PC, *all vary together*, that is, when P is free to take up any position whatever,

PA + PB + PC is a minimum,

when PA, PB, PC, make equal angles with one another,

that is, when each of the angles at P is one-third of four right angles. Q E F

From the above Prop we may infer the method of solving such questions —

Consider only two of the quantities to vary, but the third to remain constant Now find under what condition the two variables will give a maximum or minimum result, according to the question In other words, find what relationship must hold good between the two variables so as to give a maximum or minimum, according to the question.

Repeat this process, as in the above Prop, and if it is seen that the mutual relationship between the two variables is the same *in each case*, we may safely infer that this relationship will hold good reciprocally, when all the three quantities are varying simultaneously

Note. If in the above Prop we describe on each side of the triangle segments containing an angle equal to one-third of four right angles, then the three arcs will obviously intersect at one point, which is the point P required

Proposition 12 Theorem.

Of all triangles inscribed in a triangle, the pedal triangle has the minimum perimeter

(*See the figure Prop* D, *page* 376)

Let DEF be any △ inscribed in ABC

We have to shew that when of all △s inscribed in ABC, DEF has the minimum perimeter, it is the pedal △ of ABC.

Consider one side DE of the △ DEF to remain fixed in position, but the other two sides EF, FD, to vary, that is, their point of intersection F to move about along the side AB

Then EF + FD is a minimum
when F is the point of reflexion of E, D, on AB.

Similarly, if FD, DE vary, but EF remains fixed,
then FD + DE is a minimum
when D is the point of reflexion of F, E, on BC

Likewise, if DE, EF vary, but FD remains fixed,
then DE + EF is a minimum
when E is the point of reflexion of D, F, on AC

Hence, when DE, EF, FD, all vary simultaneously, that is, when D, E, F, are free to take up any position along the sides of the △ ABC,

then DE + EF + FD is a minimum
when each of the points D, E, F, is the point of reflexion of the other two on that side of the △ ABC on which it lies; in other words, when DEF is the pedal △ of ABC.

For, by Prop *D.* p 376, the sides of DEF make equal angles with the altitudes, and therefore make equal angles with the sides of ABC (since the complements of equal angles are equal). Q E D

NOTE. If a ray of light emanate from one of the points D, E, F, it will return to it after successive reflexions at the other two points, from either direction, and continue in the same path always.

EXERCISE.

Shew that in the above Prop the perimeter of the pedal triangle DEF is less than twice the length of any altitude of the △ ABC.

Proposition 13 Theorem

Of all triangles that may be inscribed in a circle, the equilateral has the maximum area

(See the figure Prop 2 page 412)

Suppose ACB to be a $\triangle$ inscribed in the $\odot$

Then if ACB has the maximum area of all $\triangle$s that may be inscribed in the $\odot$, we have to shew that ACB is equilateral.

Consider the side AB to remain fixed, but the sides AC, BC, to vary

Then by the above Prop 2, the $\triangle$ becomes a maximum when the tangent at C is parallel to AB

Similarly if the side BC remain fixed, but the sides AC, AB vary,

then the $\triangle$ becomes a maximum when the tangent at A is parallel to BC.

Likewise, if the side CA remain fixed, but the sides AB, BC, vary,

then the $\triangle$ becomes a maximum when the tangent at B is parallel to CA

Hence, if AB, BC, CA, all vary simultaneously,

then the $\triangle$ ABC is the maximum when the tangent at each vertex is parallel to the opposite side,

that is, when the $\triangle$ ABC is equilateral. [III. 32.

Q. E. D.

RECAPITULATION

We shall now sum up the whole Theory of Maxima and Minima It will be seen that almost all questions on this subject fall under one or other of the three classes mentioned above

In questions falling under *Class A*, the following laws hold good :—

[I] **On each side of a position of Maximum (or Minimum) value there are positions of equality**—not necessarily at equal distances from the Maximum (or Minimum) value, but **somewhere** in the neighbourhood.

[II] **The position of Maximum (or Minimum) value is nothing else than the position of coincidence of the two equal values mentioned in [I].**

[III] ***In those cases in which there are more than one Maximum or Minimum, the Maximum and Minimum positions occur alternately*** In other words, ***two Maximum values cannot occur in succession, but must have a Minimum value between. Similarly two Minimum values cannot occur in succession, but must have a Maximum value between.***

In questions falling under *Class B*, where Maximum or Minimum values are to be found of quantities which involve *two variables*, **the method of procedure is to reduce the expression involving the two variables into an expression involving a single variable, and then to find under what condition this single variable gives a maximum or minimum result.**

In questions falling under *Class C*, where Maximum or Minimum values are to be found of quantities which involve *three variables*, **the method of precedure is to consider only two of the latter to vary but the third to remain constant, and then to find under what condition the two variables will give a Maximum or Minimum result, by using the method of** *Class B*—In other words, **to find what relationship must hold good between the two variables so as**

to give a Maximum or Minimum, according to the question. This operation must be performed three times, considering by turns one of the three variables to be constant, but the other two to vary. If the mutual relationship between the two variables is the same in each case, we may safely conclude that this relationship will hold good reciprocally, when all the three quantities vary simultaneously.

EXAMPLES IN MAXIMA AND MINIMA

Class A.

1 Of all lines passing through either point of intersection of two circles, that whose segments intercepted in opposite directions (between the point and the circumferences) contain the rectangle of maximum area, is that which makes equal angles with the circles

2 The square is the maximum rectangle that can be inscribed in a circle

3 Hence shew that the maximum rectangle inscribed in a semi-circle is that whose larger side is double the shorter side

4 A and B are two points within a circle Find the position of the point P on the circumference such that if PAC, PBD, be drawn to cut the circle in C, D, the chord CD shall be a maximum

5 A is a point within a circle find a point P on the circumference such that the tangent at P shall make a minimum angle with PA

Class B.

1 If a point be taken within a square, and straight lines be drawn from it perpendicular to the sides, the sum of the squares on these lines is a minimum when the point is the centre of the square

2. Find the point on the hypotenuse of a right-angled triangle the sum of the squares on whose distances from the sides is a minimum, also, the point which gives a maximum

3. Hence find the point on the circumference of a circle the sum of the squares on whose distances from two fixed straight lines at right angles (whether they intersect within or without the circle) is a minimum, also the point which gives a maximum.

Class C.

1 Of all triangles of an equal area, the equilateral has the minimum perimeter

2 Of all triangles of an equal perimeter, the equilateral has the maximum area

3 Find a point within a triangle such that the sum of the squares on its distances from three angular points may be a minimum

4 Within an equilateral triangle inscribe an equilateral triangle of minimum perimeter

5 Of all equilateral triangles that which has the greater perimeter has the greater area, and *vice versa* Hence, in the last Ex shew that the inscribed equilateral triangle has also the minimum area

6 Hence describe the maximum equilateral triangle (as regards perimeter and area) about a given equilateral triangle.

7 Similarly, in a given square inscribe the minimum square (as regards perimeter and area)

8 Hence, describe the maximum square (as regards perimeter and area) about a given square

———o———

HINTS AND SOLUTIONS.

Prop 4

1 Let PQ (*produced*) cut the ⊙ at M, M'; and let C and C' be the points of contact of the two ⊙s which pass through P, Q, and touch the given ⊙ (in both cases the ⊙s touching *externally*) Then the points C, C', give maximum, and M, M', minimum angles, also, these four points lie on the circumference in the following order :—C, M, C', M'.

If the line joining P and Q cut the ⊙, then both the described ⊙s touch the given ⊙ internally, hence these points of contact give *minimum* angles, and the points where PQ cuts the given ⊙, give maximum angles (two right angles)

2 The points of intersection M, M', now coincide, hence there will be only *one* minimum angle (if PQ produced is a tangent) Also, in this case only one ⊙ can be described to pass through P, Q, and touch the given ⊙, hence there will be only *one* maximum angle.

If the line joining PQ is a tangent; then the former point gives a maximum and the latter a minimum.

3 This is much the same as Ex 1 Let the line joining PQ, when produced both ways, cut the ⊙ at M, M', and let C, C', be the points of contact of the two ⊙s which pass through P, Q, and touch the given ⊙ (in both cases the described ⊙s touching the given ⊙ *internally*) Then the points C, C', give maximum, and M, M', minimum angles, also these four points lie on the circumference in the following order —C, M, C', M'

4 Since the line joining P, Q, when produced both ways, must cut the given ⊙ at two points (say M and M') there will always be two minimum angles But two minimum angles cannot occur in succession without a maximum angle intervening . in each arc (formed by the secant MM') there must be a maximum angle Hence in whatever relative position P, Q, are placed within the ⊙, there must always be *two* maximum (besides the two minimum) angles In other words *two* ⊙s can always be drawn which pass through two internal points and touch a given circle

Prop 5.

1 Suppose PCB to be the required line, cutting AC at C and AB at B

Let there be two lines through P on either side of PCB (one of them cutting the arms at D, E, and the other cutting the arms at G, F,) so that the rect $DP\ PE$=the rect $GP\ PF$ Then D, G E, F, are cyclic (Prop V p 349) Hence AB, AC, are secants to that ⊙, hence AB, AC, become tangents to the ⊙ whose common chord is BC, etc as in the above Prop

2 Let BPC be a position of minimum as found in the Proposition Now suppose the straight line BPC to revolve about the point P in either direction, say to the right (as the hands of a watch)

Then as the point B travels towards A, the point C will cut AC more to the left till when PB becomes parallel to the arm CA, the point C, where BP intersects this arm, goes off to an infinite distance Hence the rect $BP\ PC$ becomes infinite, and consequently the line BP (parallel to the arm AC) gives a maximum result.

Now if B comes still nearer to A, then PB (produced) must cut CA produced—at a very great distance at first, then nearer and nearer to A, as B also comes nearer to A Therefore the rect $BP\ PC$ from being infinite, becomes smaller as the revolving line approaches the angular point A, till it again becomes a minimum when the line again makes equal angles with one of the arms and the other produced

Hence, there are two positions of minimum—the first when the line through P makes equal angles with the arms, and the second when the line through P makes equal angles with one arm and the other produced

Also, there are two positions of maximum—when the line through P is parallel to either arm

And obviously from the above, these four cases occur alternately

3. Proceed exactly as in the last Ex

Prop 6

It is obvious that the ends of the chord give minimum results, and the ends of the diameter at right angles to the chord give maximum

Prop 7

1 Apply Prop III p 187, and Euc III 8

2 Apply Prop III p 187, and Euc III 7

3 The two points must be so situated that the line joining its middle point with every point on the circumference is constant Hence the middle point of the line joining the fixed points is the centre of the ⊙, ∴ the fixed points are the ends of a diameter

4. Let E be the middle point of AB Then *wherever P may be*, $AP^2+BP^2=2AE^2+2EP^2$ ∴ $AP^2+BP^2=$a constant$+2EP^2$. Hence AP^2+BP^2 is the least possible when $2EP^2$ is the least possible, that is, when $2EP^2$ is nothing that is when P coincides with E Hence the least possible value of $AP^2+BP^2=2AE^2$

5 The line must obviously be bisected, since this is only a particular case of the last Ex when P is any point on the line AB And since the last Ex is true universally, it is also true in this case

Prop. 12.

Take any altitude CF. A triangle may be imagined to be inscribed in ABC, of which one vertex F would lie on AB, and the vertex on AC would lie very close to C, and the vertex on BC would also lie very close to C Hence the perimeter of this triangle would be equal to twice CF, and must be greater than the perimeter of DEF by the above Prop Hence DEF would be less than twice CF.

Class A

2 Let $ABCD$ be any rectangle inscribed in a ⊙. The diagonal AC is a diameter ∴ the △ ABC is a maximum when $AB=BC$, etc

4. If CD be a maximum, the angle CPD is also a maximum; etc

5 The minimum angle at P gives the minimum angle in the alternate segment ; etc

Class B

2 Let $\triangle$ ABC be right-angled at B, let P be a point on AC, PM, PN, perps to AB, BC, respectively Then, $PM^2+PN^2 = MN^2 = BP^2$

Hence PM^2+PN^2=a minimum, when BP is a minimum, that is, perp to AC

Also, it is a maximum, when BP is a maximum, that is, when BP coincides with the longer side

3 As in the last Ex the required point for a minimum is that whose distance from the fixed lines is a minimum , similarly for a maximum Apply Euc III 7 or 8

Class C

3 Let ABC be the $\triangle$ It is required to find the point P within the $\triangle$, so that $PA^2+PB^2+PC^2$=a minimum Suppose PA^2 to be constant, that is, PA to be constant, that is, P to lie on the circumference of a ⊙, whose centre is A Let D bisect BC. Then PB^2+PC^2=a *minimum* when DP is a *minimum*, that is, when DP produced passes through the centre A (III 8), that is, when P lies on the median AD Hence, when P is free to move anywhere within the $\triangle$, $PA^2+PB^2+PC^2$=a minimum when P is the centroid of ABC

4. Of all $\triangle$s inscribed in the given $\triangle$, the pedal has the minimum perimeter But in this case the pedal $\triangle$ is also equilateral ∴ the pedal $\triangle$ is the required one

5 The equilateral $\triangle$ which has the greater perimeter, has the greater side and also the greater altitude , and ∴ it has the greater area.

6 The described $\triangle$ will be a *maximum*, when the given $\triangle$ is a *minimum* with respect to it (from the last two Exs), that is when the given $\triangle$ is the pedal $\triangle$ of the described one , in other words, when the described $\triangle$ is the antipedal of the given $\triangle$.

7 and 8 The method is the same as in Exs. 4, 5, and 6, although in the case of the perimeter there are *four* variables in these two Exs Consider first the perimeter (*i e* find the points of reflexion) and then deduce for the area

APPENDIX.

CALCUTTA ENTRANCE EXAMINATION PAPERS.

1858-1859

1 If one of the acute angles of a right-angled triangle be double of the other, the hypotenuse is double of the shorter side.

2 If any point be taken within an equilateral triangle, the sum of the perpendiculars drawn from it to the sides is equal to the perpendicular from the vertex to the base

3 Show that the diagonals of a rhombus bisect one another and cut at right angles what propositions do you assume in your proof

4 In any triangle ABC, if the angles at A and B be bisected by straight lines which meet in D, show that the line joining D and C will bisect the angle ACB

5 The squares on the diagonals of a parallelogram are together equal to the sum of the squares on the four sides.

1859

1 From the same point there cannot be drawn more than two equal straight lines to meet a given straight line

2 Prove that the four triangles, into which a parallelogram is divided by its diagonals, are equal to one another

3 If two chords of a circle intersect at right angles, the portions of the circumference, taken alternately, are together equal to half of the circumference

4. If two circles cut one another, find a point from which the straight lines drawn to touch the two circles shall be equal to one another

1860

1 If, in Fig Prop 5, Book I, H be the point of intersection of BG and CF, prove that AH will bisect the angle BAC

2. From a given point draw a straight line making equal angles with two given straight lines.

3 If on the radius AO of a circle whose centre is O, a semicircle be described, and from any point in AO, a straight line be drawn at right angles to it, cutting the semicircle at P and the larger circle at Q, and if AP and AQ be joined, show that the square on AQ will be double of the square on AP.

4 Any angle of a triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained by the side subtending the angle, and a diameter from either extremity of that side

1861

1. Through a given point draw a straight line which shall make equal angles with two straight lines given in position

2. If the straight line bisecting the vertical angle of a triangle, also bisect the base, the triangle is isosceles

3 The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals

4 Given the angle at the base of an isosceles triangle, and the perpendicular from it on the opposite side, construct the triangle.

1862

1 Construct an isosceles triangle whose exterior vertical angle is $67\frac{1}{2}$ degrees

2 In the side BC of a right-angled triangle ABC, right-angled at C, find a point D, such that the perpendicular DF drawn from D to a point F in the hypotenuse shall be equal to AF.

3. The area of a rhombus is equal to half the rectangle contained by the diagonals.

4 Given a chord AB of a circle and a point C in it; find in the circumference a point D, such that the line DC shall bisect the vertical angle of the triangle ABD

1863

1 Given two equal and parallel straight lines AB and DC: prove that AC and BD bisect each other Under what circumstances will AC equal BD?

2 Three straight lines meet in a point Draw another line cutting them, so that the segment of it intercepted between the first and second, shall be equal to that intercepted between the second and third.

3. Describe a square that shall be equal to a given triangle.

4 What is the locus of the middle points of equal straight lines in a circle?

5 A tangent is drawn parallel to a chord. Show that the intersected arc is bisected at the point of contact.

1864

1. Show that every four-sided figure, whose opposite sides are equal, is a parallelogram

2 In a right-angled triangle, the line joining the right angle and the point of bisection of the hypotenuse, is equal to half the hypotenuse.

1865.

1. Resolve *any one* of the following :—

a Given one of the sides of a right-angled triangle containing the right angle and the sum of the other two sides, to construct the triangle

b. Given one of the sides of a right-angled triangle containing the right angle and the difference of the other two sides, to construct the triangle

c The straight line drawn from the right-angle of a right-angled triangle to the middle of the opposite side, is equal to half of that side

2. Resolve *either* of the following —

a Divide a given straight line into two parts, so that the rectangle contained by them, shall be equal to a given square

b Produce a given straight line, so that the rectangle contained by the whole line thus produced and the part of it produced, shall be equal to a given square

3 Demonstrate *either* of the following —

a If a rectilineal figure of an even number of sides be inscribed in a circle, the first, third, fifth, &c angles are together equal to the second, fourth, sixth, &c , angles taken together, any angle being assumed as the first

b If a circle be inscribed in any triangle, the points of contact shall divide the sides into segments, such that any one side together with the remote segment of either of the other two sides, shall be equal to half the sum of the three sides.

1866.

1. AB is parallel to CD and unequal to it, and they are joined towards the same parts by the straight lines AC and BD. If AC is equal to BD, show that AD is equal to BC.

2. Describe a circle which shall touch a given straight line at a given point, and pass through another given point.

3. Produce a given straight line to a point such that the rectangle contained by the whole line thus produced and the part produced, shall be equal to the square on the given straight line.

4 ABC is an isosceles triangle of which B is the vertex; BA, BC are bisected in D and E respectively, AE, CD, intersect at F. Show that the triangle BDE is equal to three times the triangle DEF

5 Construct a rectangle that shall be equal to a given square, the difference of two adjacent sides being given

6. If a tangent of a circle be parallel to a chord, prove that the intercepted arc is bisected at the point of contact of the tangent.

7. To describe a circle that shall touch a given line and also touch a given circle

1867

1 Construct an isosceles triangle having each of the sides double of the base

2. The straight line which bisects the vertical angle of an isosceles triangle, bisects the base perpendicularly.

3. Describe a rhombus equal to a given square

1868.

1. Prove that the three interior angles of every triangle are equal to two right angles, *without producing a side of the triangle.*

2. Show from I. 47, how to find a square which shall be equal to the difference of two given squares

3 Prove by means of II 12 and II 13 that if any side of a triangle be bisected, the squares on the other two sides are together equal to twice the square on half the line bisected, and twice the square on the line drawn from the point of bisection to the opposite angle.

4 The three points of contact of a circle inscribed in a triangle are joined, show that the resulting triangle is acute-angled.

5 Perform *one only* of the following deductions —

(1) Construct a right-angled triangle, having given the hypotenuse and the sum of the sides

(2) Two circles have the same centre, show that all chords of the outer circle which touch the inner circle are equal.

1869.

1 Having given the base of a triangle, the difference of the sides, and the difference of the angles at the base , it is required to describe the triangle.

2 If two circles intersect one another, their common chord, when produced, bisects their common tangent

3 Inscribe a circle in a rhombus

1870

1 Given that two triangles are between the same parallels and equal in area , prove that their bases are equal

2 Two straight lines OA, OB being given, intersecting in O, and a point C being given in OA describe a circle touching OA in C, and also touching OB

3 The straight line drawn from the right angle, in a right-angled triangle, to the bisection of the hypotenuse, is equal to half the hypotenuse.

1871.

1 In a polygon of n sides the sum of all the internal angles equals $(2n—4)$ right angles

2 Find a line whose square shall be equal to the sum of the squares on three given right lines

3. ABC is a right-angled triangle. AD, the perpendicular from A upon the hypotenuse BC, is produced in the direction DA, till it meets a side produced of the square on AC in O. Prove that it will meet a side produced of the square on AB in the same point O, that AO shall be equal to BC, and that if O be joined with B, and A with E the extremity of the side BE of the square on BC, the figure $OAEB$ shall be a parallelogram equal in area to the square on AB.

4. Given the base, the vertical angle, and the perpendicular let fall from the vertex on the base, construct the triangle and show that *in general* there can be two triangles constructed satisfying the given conditions.

5. The perpendiculars erected at the middle points of the sides of a triangle, meet in a point.

1872.

1. Divide a line so that the rectangle contained by the parts shall be the greatest possible.

2. In a triangle APB, the square on AP is less than the square on BP by a constant quantity. Prove that P must be on a certain straight line.

3. If the parts of two chords at right angles to one another be given, explain how the length of the radius of the circle may be calculated.

4. Compare the area of a regular hexagon inscribed in a circle, with that of an equilateral triangle inscribed in the same circle.

5. Express each of the angles of a regular pentagon and also of a regular quindecagon in terms of a right angle.

1873.

1. If the middle points of the three sides of a triangle be joined, the triangle so formed shall be equiangular to the given triangle and equal to one fourth of it.

2. The exterior angles DBC and ECB of the triangle ABC are bisected by BF and CF, FG and FH are drawn perpendicular to AD and AE, prove that FG is equal to FH, and AG to AH.

3 AB is a chord of a circle, C a point in the circumference of the smaller segment, find a point D in the circumference of the larger segment so that AB shall bisect the angle DBC

1874.

1. Deduce from Euc Book II, Prop 4, that the square on the whole line is four times the square on half the line.

2 Prove that if an angle of a triangle be two-thirds of a right angle, the square on the side opposite to it is equal to the sum of the squares on the sides containing it, diminished by the rectangle contained by them.

3. State, *without proving*, the conditions which must be fulfilled in order that a circle may be described so as to pass—(1) through two given points, (2) through three given points, (3) through four given points

4. A circle is described so as to touch the side BC of the triangle ABC in D, AB produced in E, and AC produced in F; show that the triangle EDF is obtuse-angled.

5. QA and QB are two straight lines in a circle at right angles to one another, QD is a diameter, P any point in the circumference of the smaller segment cut off by QA : show that the area of the triangle APQ together with the area of the triangle BQP is equal to the area of the triangle QPD.

1875.

1 If two straight lines are equal and parallel, show that if the extremities be joined but not towards the same parts, two equal triangles will be formed.

2. Show how the enunciation of the II. 9 (if a straight line be divided . . the squares on the two unequal parts are together double . .) may be made to include II. 10, and prove both propositions as one

3. Show that if from any point without a circle, straight lines be drawn touching it, the angle contained by the tangents, is double the angle contained by the straight line joining the points of contact and the diameter through either of them

4 Prove the following rule —" Divide the square of the chord of half the arc by the height of the arc, and the quotient will be the diameter of the circle"

1876

1 AB and CD are two straight lines intersecting at O, CA and DB are perpendiculars to AB OB is double of OA Prove, without making use of the properties of similar triangles, that OD is double of OC

2 C is the centre of a given circle, A any other point within it AB is drawn at right angles to AC and meets the circumference in B. Prove that the circle described about the triangle ABC touches the given circle, and that ABC is the greatest angle subtended by AC at any point in the circumference of the given circle

3 AB and AC are chords of a circle at right angles to one another, their lengths are 30 feet and 40 feet respectively Find the height of the arc AC and the diameter of the circle.

1877

1. Draw a common tangent to two given circles.

2. State what regular polygon has each of its angles equal to nine-tenths of two right angles

1878.

1. The two sides of a triangle are 9 and 12 feet respectively, the angle contained by them is equal to the other two, find the length of the third side

2. Describe a circle touching one side of a triangle and the other two produced.

3 ABC is a triangle with a right angle at A, AD is perpendicular to BC, to what rectangle is the square on AD equal ?

4 $ABCD$ is a quadrilateral whose area is 8,575 square yards, B is a right angle BL is perpendicular to AC, AL is 90 yards and CL 40 yards · find the area of CAD. P is the middle point of CD, and PQ is parallel to CA · find the length of PQ

5 The sides of a triangle are 8, 6 and 10 feet respectively : find (*a*) its area, (*b*) the diameter of the circumscribing circle, (*c*) the height of the arc cut off by the side 8 feet in length.

1879

1 In two circles which touch each other externally, two parallel diameters are drawn Show that one extremity of each diameter and the point of contact lie in the same straight line.

2 A circle is described to touch BC, a side of the triangle ABC, in D, and the other two sides produced in E and F respectively Prove that AF is equal to one half of the sum of the sides of the triangle ABC

3 Two fixed points A and B lie on the same side of a fixed straight line CD of unlimited length P is any point in CD Prove that the sum of the lengths AP and BP is least when the angles which AP and BP make with CD are equal.

1880

1 The side BC of the triangle ABC is produced to D, show that the angle ACD is greater than the angle ABC without showing that it is greater than the angle BAC.

2 AOC, BQD are two triangles having the angle AOC equal to the angle BQD, and the angle ACO equal to the angle DBQ, show that the rectangle contained by AO and QB is equal to that contained by CO and QD.

3 Show that the square on the side of an equilateral triangle described about a circle is four times the square on the side of an equilateral triangle inscribed in the same circle.

1881

1. Show how to make a triangle equal to a given quadrilateral which shall have its base on one side of the quadrilateral produced if necessary, and its vertex at one of the opposite angles.

2. BC is a given arc of a circle whose centre is O; A is any point in BC AD, AE are drawn perpendicular to OB, OC. Prove that the line DE is of constant length.

1882.

1 OC is a straight line which bisects the angle AOB, and OD is any other straight line without the angle AOB, show that the angles DOA, DOB are together double of the angle DOC

2 ABC is a triangle, straight lines AD, CE bisect the angles at A and C, and from B, BE is drawn equal to BC, and BD equal to BA, show that EBD is a straight line.

3 If A, B be fixed points, and O any other point, the sum of the squares on AO and BO is least when O is the middle point of AB

4 If two straight lines AB, CD in a circle intersect in E, the angles subtended by AC and BD at the centre are together double of the angle AEC

5 AO, BO are radii of a circle at right angles to each other. ACD is a straight line meeting OB in C and the circle in D. Then the rectangle contained by AC, AD is double of the square on OB

1883

1 ABC is a triangle The line bisecting the angle B meets the line bisecting the angle C in the point G, and the line bisecting the external angle at A in the point D Prove that the angle ADG is equal to the angle ACG

2 Through one extremity of the common chord of two intersecting circles, two straight lines are drawn terminated by these circles. Prove that the lines joining the other extremity of the common chord and the two terminal points of the two straight lines on each circle, together with the lines joining these terminal points, form two equiangular triangles

1885.

1 Divide a given straight line into two parts, such that the difference between the squares described upon the two parts may be equal to the square on a given straight line

2 AB is a diameter of a circle, and AC a tangent at A equal in length to AB, CB is joined cutting the circle in D; prove that CB is bisected in D, and AD equal to half of CB.

3. Describe a circle touching three given straight lines lying in one plane, no two of which are parallel. Show that *four such* circles can be described.

4 ABC is an acute-angled triangle. Perpendiculars AD, BE, CF, are drawn from the angular points A, B, C, upon the

opposite sides respectively, intersecting in O, prove that O is the centre of the circle inscribed in the triangle DEF, and A, B, C are the centres of circles *escribed* to the same triangle.

1886

1. If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram

2. The square described on the difference of two straight lines together with twice the rectangle contained by the two lines is equal to the sum of the squares described on them

3 (*a*) If two circles touch internally, the centre of the interior circle lies in that radius of the exterior circle which passes through the point of contact

(*b*) Also, show that any chord of the exterior circle drawn from the point of contact is bisected by the interior circle, if that circle passes through the centre of the exterior circle.

4 With the aid of an isosceles triangle such that each of the angles at its base is seven times the angle at the vertex, to inscribe a regular quindecagon in a given circle Give the geometrical proof

5. If the middle points of the three sides of a triangle be joined to the opposite angles by three straight lines, prove that the sum of these three lines is less than the sum of the three sides.

6. Two equal circles intersect in A and B Let CD and EF be chords of the circles, each equal to the chord AB, and so placed on opposite sides of AB, that all the three chords meet in H Then AH bisects the angle CHE.

1887.

1. $ABCD$ is a quadrilateral of which the sides AB and DC are parallel, and E, F are the middle points of the sides BC and AD respectively, prove that the straight line EF is parallel to AB or CD and equal to half their sum.

2 If two circles cut each other, their common chord produced bisects their common tangents

3. AB, AC are tangents to a given circle, and BC is the chord joining the points of contact From the middle point D of BC, the straight line EDF is drawn at right angles to BC cutting the circumference of the given circle at E and F. Prove that E and F are the centres of two circles, one of which touches the three sides, and the other touches one side and two sides *produced*, of the triangle ABC.

1888.

1 Prove that if the middle points of the sides of a quadrilateral be joined, the figure formed is a parallelogram whose area is equal to half that of the quadrilateral.

2 Prove that if from a point two straight lines be drawn to touch a circle, these straight lines are equal.

3. *ABC* is a triangle; *DEF* a straight line meets the side *AB* at *D*, *BC* at *E* and *AC* produced at *F*, the point of intersection of the circles circumscribed about *DBE* and *ECF* besides *E* is *G* Prove that the circles circumscribed about *ABC* and *ADF* also pass through *G*.

1889.

1 Divide the hypotenuse of a right-angled triangle into two parts, such that the difference between their squares shall be equal to the square on one of the sides

2 Construct a square equal to a given equilateral triangle.

3 Describe a circle about a given triangle, and show that the three perpendiculars dropped from the vertices on the opposite sides of any triangle meet in a point

1890

1. In a right-angled triangle the line joining the right angle to any point (except the middle point) of the hypotenuse, is greater than one segment of the hypotenuse and less than the other

2 Prove that the area of a quadrilateral is equal to the area of a triangle having two sides equal to the diagonals of the quadrilateral, and the contained angle equal to that between the diagonals.

3. (In fig II 11)—Prove that the rectangle contained by the two parts is equal to the difference of the squares on the two parts

4. From a given point without the circumference of a given circle, show that two and only two tangents can be drawn.

5. If two opposite sides of a quadrilateral inscribed in a circle are equal, prove that the other two sides are parallel

6. *P* is a point in *APB* an arc of a circle. The tangent at *P* meets the chord *AB* produced in *R*, and *AQ* perpendicular to *AB* in *Q*; and *RQ* is bisected in *P*. Prove that the angle *ABP* is double of the angle *BAP*.

1891.

1. Define a plane angle, the centre of a circle, parallel straight lines, the angle *of* a segment, and an angle *in* a segment.

2. If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle

Prove the necessity of the condition that the lines are to be drawn from the *ends* of the side

3 In a right-angled triangle, the square described on the hypotenuse is equal to the sum of the squares described on the other two sides

4 In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle

5 Let B and C be two fixed points, and PQ a straight line in the same plane as B, C Find the position of the point A on the straight line PQ, which is such that the sum of the squares on AB, AC is least

6 If a straight line drawn through the centre of a circle, bisects a straight line in it which does not pass through the centre, it shall cut it at right angles, and if it cut it at right angles, it shall bisect it

7 If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent shall be equal to the angles in the alternate segments of the circle

8 Draw a common tangent to two circles, and show that, in general, four common tangents may be drawn to two given circles.

9. Give *only* the constructions of—

(*a*) IV 4. To inscribe a circle in a given triangle.

(*b*) IV. 10. To describe an isosceles triangle having each of the angles at the base double of the third angle

10. In the triangle ABC, O is the centre of the inscribed circle, and O_1, O_2, O_3 the centres of the escribed circles (that is, circles touching any side and the other two sides produced). Show that the four circles, each of which passes through three of the points O, O_1, O_2, O_3 are all equal.

1892

1 Define a plane surface, a rhombus, and an axiom. What axiom affords the ultimate test of equality of two geometrical magnitudes?

2 Prove that on the same base and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another

3 At a given point in a given straight line you are required to make an angle equal to a given angle

4 If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided line

5 You are required to find the centre of a given circle

6. Prove that one circle cannot cut another at more than two points

7 You are required to inscribe a regular quindecagon in a given circle

(*a*) Show that if a polygon inscribed in a circle is equilateral it is also equiangular

8. Bisect a quadrilateral figure by a straight line drawn through an angular point

9 Describe a circle to touch a given circle and also to touch a given straight line at a given point

10. Prove that of all triangles of given base and area, the isosceles is that which has the least perimeter

1893.

1 Define a right angle, a rectangle, a tangent to a circle, and a regular polygon.

2 If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely those to which the equal sides are opposite, that is to say, the triangles shall be equal in all respects Prove this proposition

3 *ABCD* is a parallelogram, and *KD*, *KB*, are the complements of the parallelograms *EH*, *GF* about the diagonal *AC*, *EKF* being parallel to *AHD*, and *GKH* to *AEB*. Show that the complement *BK* will be equal to the complement *KD*.

Show also that the gnomon *BHF* will be double the triangle *CEH*.

4. In an obtuse-angled triangle, if a perpendicular is drawn from either of the acute angles to the opposite side produced, show that the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the line intercepted without the triangle, between the perpendicular and the obtuse angle.

5 Prove that the angle at the centre of a circle is double of an angle at the circumference, standing on the same arc.

6 If from any point without a circle a tangent and a secant be drawn, prove that the rectangle contained by the whole secant and the part of it without the circle will be equal to the square on the tangent

7 In a given circle inscribe a triangle equiangular to a given triangle.

8 Bisect a triangle by a straight line drawn through a given point in one of its sides

9 Two circles touch each other externally in A, and a straight line touches them in B and C respectively Prove that BAC is a right angle

10. Given the base and vertical angle of a triangle, find the locus of the centre of the inscribed circle

1894

1 From a given point in a given straight line you are required to draw a straight line equal to the given straight line.

2 Define parallel lines Define a plane

Prove that if a straight line falling on two other straight lines make the alternate angles equal to one another, then the straight lines shall be parallel

3 You are required to construct a square equal to a regular pentagon.

4 Prove that if any two points are taken in the circumference of a circle the chord which joins them falls within the circle.

5. Prove that similar segments of circles on equal chords, are equal to one another

6. From a given circle you are required to cut off a segment which shall contain an angle equal to a given angle.

7. By the fourth Book of Euclid you are required to construct an angle equal to the one-thirtieth part of a right angle

8. Trisect a right angle.

9. Describe a circle passing through two given points and touching a given straight line

10 Given two points A and B and a right line L, find a point P in L such that AP plus BP shall be a minimum.

Allahabad Entrance Examination Papers.

1889

1 Enunciate all the propositions of Euclid, Book I, in which the equality of three parts in a pair of triangles involves equality in all respects.

2 Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides

3 To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle

4 From a given point in one of the sides of a triangle, draw a straight line to meet the other side produced, so that the triangle thus formed shall be equal to the given triangle

5 In every *obtuse angled* triangle, the square on the side subtending an *acute* angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle

6 In any triangle, the sum of the squares on the two sides, is equal to twice the square on half the base, together with twice the square on the straight line joining the vertex to the middle point of the base

7 (*a*) From a given circle to cut off a segment containing an angle equal to a given rectilineal angle

(*b*) Having given the base and the vertical angle of a triangle, show that the triangle is greatest when it is isosceles

8 To inscribe an equilateral and equiangular pentagon in a given circle

1890

1 Define a straight line, an acute-angled triangle, a circle, parallel straight lines, a gnomon, an angle in a segment.

When are magnitudes said to be equal? What is meant by the height of a triangle?

2. If two triangles have two sides of the one equal to two sides of the other each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.

(A direct proof of this proposition may be given if preferred.)

3. Prove that if a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another.

4. If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.

5 If the square described upon one of the sides of a triangle be equal to the squares upon the other two sides, show that the angle contained by these two sides is a right angle.

6. Given that the square on a line divided into any two parts is equal to the squares on the two parts together with twice the rectangle contained by the parts , from this proposition deduce a proof of the 47th proposition of Book I.

7 If a straight line be divided into any two parts, the squares on the whole line and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part

8 The angle at the centre of a circle is double of the angle at the circumference upon the same base

9 If two straight lines cut one another within a circle neither of which passes through the centre, prove, without assuming the previous cases proved by Euclid, that the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other

10 To describe an isosceles triangle having each of the angles at the base double of the third angle

Give the *construction only* of this problem.

(*b*) Divide a right angle into five equal parts.

1894

1. Define a *parallelogram*, a *gnomon*, an *arc*, a *segment* of a circle

2. Euc I. 8 3 Euc I. 44. 4. Euc. II. 7.

5 Euc. III. 15. 6 Euc IV 5.

7 Inscribe a square within an equilateral triangle.

8. In a circle two chords AEB and CED intersect at E. Prove that the angles subtended by AC and BD at the centre are together double of the angle AEC.

9. Prove the formula for determining the radius of the circle inscribed in a triangle whose sides are given.

Riders in Punjab Entrance Papers.

1. What is the difference between the angle *of* a segment and the angle *in* a segment of a circle? Prove that the centre is the only point in a circle at which two chords can bisect each other

2 If in any triangle the straight line which bisects the vertical angle also bisects the base, the triangle is isosceles

3 ABC is a triangle right-angled at A, and AD is drawn perpendicular to the base BC, shew that the square on AB is equal to the rectangle contained by BC, BD

4 Divide a given straight line into two parts, so that the rectangle contained by its segments may be equal to a given square.

5 Shew how to trisect a given straight line

6 Prove that the difference between two of the sides of any triangle is less than the third side

7 Show that the difference of the base angles of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base and another bisecting the angle at the vertex

8 The perpendiculars let fall from the three angles of any triangle upon the opposite sides intersect each other in the same point

9 Find the locus of a point from which tangents drawn to two circles are equal (1) when the circles touch each other externally, (2) when they do not

10 Inscribe an equilateral triangle in a circle, and compare its area with that of a regular hexagon inscribed in the same circle

11 Two circles touch internally at A A straight line touches one circle at P, and cuts the other at Q and R Prove that PQ and PR subtend equal angles at A

12 From the 13th Proposition of the 2nd Book Euclid, deduce an expression for the area of a triangle in terms of its sides.

13 Give geometrical demonstration of $(x+a)^2-(x-a)^2 =4ax$.

14 If AB, AC be equal sides of an isosceles triangle, and a circle with centre B and distance BA cut AC (or AC produced) in E, and BF be taken in AB (or AB produced, if E lies

in AC produced) equal to CE, prove that the angle CFA is equal to the angle FAC.

15 Prove that the difference between the squares described upon two straight lines is equal to the rectangle contained by their sum and difference

16 If two straight lines be drawn through any point on a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides will be on the circumference of a circle whose centre is at the intersection of the diagonals

17 Having given one side of a triangle, and the centre of the circumscribed circle, determine the locus of the centre of the inscribed circle.

18 Draw lines through the angular points of a parallelogram which shall form another parallelogram equal to twice the former.

19 Having given the sides of a triangle, to find the diameter of the circle described round the triangle

20 All the exterior angles of a pentagon made by producing the sides successively in the same direction are together equal to four right angles

21 In the diagram given by Euclid in II 11 point out any other line, besides the given one, similarly divided.

22 The straight lines drawn from the angles of a triangle to the points of bisection of the opposite sides meet at the same point

23. DR is a diameter of a circle, DP, DQ are two chords, meeting the tangent at R, at S and T respectively Show that the angles TPS and TQS are equal

24. Find the circumference of the circle described round a triangle whose sides are 21, 17 and 10 feet.

25 Prove that if two circles touch one another, the circumferences cannot have a common point out of the direction of the straight line which joins the centres What different cases are there of this proposition ?

26 Find a square that shall be equal to the difference of two given squares.

27. Construct a triangle having given an angle and the radii of the inscribed and circumscribed circles.

28. "The area of a parallelogram is equal to the product of the base and the perpendicular height." Prove this.

29. One acute angle of a right-angled triangle is double the other, show that the side opposite to the less is equal to half the hypotenuse.

30 Divide a right line into 2 parts such that the rectangle contained by the parts may be the greatest possible

31. AD is drawn perpendicular to the base BC of a scalene triangle ABC Prove that $AB^2+BC^2=AC^2+2BC\,BD$

32. Hence deduce an expression for the area of a triangle in terms of its sides, and show what form this expression assumes when the triangle is right-angled

33. What relation subsists between an angle *in* a segment of a circle, and an angle *on* the same segment ?

34. Divide a circle into two segments so that the angle in one segment may be double that in the other

35 Shew by computation that a side of the equilateral triangle together with a side of the square, both inscribed in the same circle, is equal to half the circumference of that circle *nearly*.

36. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremeties, and a third tangent Show that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

37 Two diameters AOB, COD of a circle are at right angles to each other, P is a point in the circumference ; the tangent at P meets COD produced at Q, and AP, BP meet the same line at R, S, respectively Show that $RQ=SQ$

38. AB is a fixed chord of a circle, AC is a movable chord of the same circle (the angle CAB being therefore variable) A parallelogram is described, of which AB and AC are adjacent sides, determine the greatest possible length of the diagonal through A.

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CALCUTTA EXAMINATION PAPERS.

Hints for Solution.

1858

1 Let ABC be the triangle right angled at B, and the angle at C double of that at A At B in CB make the angle CBD equal to the angle ACB, meeting AC at D Apply Euc I 6, 32

2 Let ABC be the equilateral triangle, and from D a point within it are drawn DE, DF and DG perpendiculars to AB, BC and CA; also let AK be the perpendicular from A on BC Through D draw LDN parallel to BC and through L draw LR perpendicular to AC Produce GD to M and draw LM perpendicular to GM Apply Euc I 34, 26

3 Apply Euc I 8 and 4.

4 The perpendiculars from D on the sides are equal Apply Euc I. 26

5 The diagonals bisect each other Apply Addl Prop III, p 187.

1859.

1 Apply Euc I. 16 and 19 2 Apply Euc I 26 and 38

3 See Ex 3 of Euc III, 26

4 Any point in the produced part of the line joining the points of intersection is one of the required points

1860

1. Apply Euc I 6, 8

2. Let the two given lines AB, AC meet at A, bisect the angle BAC by AD. From E the given point draw EP perpendicular to AD Produce PE to meet AB, AC at F, G respectively.

3. Apply Euc. III. 31, and Addl. Prop II *Cor*. p. 186.

4. Apply Euc III 31, and 21

1861.

1. See Ex. 2. 1860

2. Let ABC be the triangle. AD bisects the vertical angle. Produce AD to E making DE equal to AD; join BE. (I. 6)

3 See Ex 5, 1858.

4 Let ABC be the given angle. At B in CB, and on the other side of it, make the angle CBD equal to the angle ABC. Draw BE at right angles to BD making BE equal to the given perpendicular Through E draw CEA parallel to BD meeting BC at C and BA at A

1862

1 Let a line AC be at right angles to AB, bisect the angle BAC by AD and bisect again the angle DAB by AE In CA produced take AF equal to AE, join EF

2 At A in BA make the angle BAD equal to half a right angle From D draw DF perpendicular to AB

3 The diagonals bisect each other at right angles (1858, 3) Apply Euc I 41

4 Bisect the arc AB at E join FC and produce it to meet the circumference at D Join AD, DB (Euc III 26)

1863

1 Apply Euc I 29, 26 AC is equal to BD when $ABCD$ is a rectangle

2 Let BA, CA, DA meet at A Take any point C in AC. From C draw CF parallel to BA, and CE parallel to FA Join FE cutting AC at G

3 Euc II 14

4 A circle is the locus, since all straight lines drawn from the centre to these points are perpendiculars on those lines (III 3), also they are equal (III 14)

5 Apply Euc III 18, 28

1864

1 Apply Euc. I 8 and 27

2. Addl Prop. IV, p 91

1865

1 a. & b See Ex 22, Book I, p 114

c Addl Prop IV. p. 91.

2 (a) See Ex. 8, Book II. p 193.

(b) See Ex. 13, Book II. p 193.

3 (a) Apply Euc. III. 22

(b) See [I] (1) p 385.

1866.

1. Through D draw DE parallel to CA meeting AB or AB produced at E BD is equal to DE. Apply Euc. I. 4.

2. See Ex 4. of Prop. 18, Book. III.

3. See Ex. 13, Book II. p. 193.

4. Apply Euc I 38. The restriction "ABC is an isosceles triangle" is not necessary.

5. Let CB be equal to half the difference Draw BD at right angles to CB making it equal to the side of the given square. Join CD. Produce CB both ways making CA, CE each equal to CD. Apply Euc. II 6

6. Apply Euc. III. 28.

7. Let AB be the given line and F the centre of the given circle Draw FEG perpendicular to AB and cutting the circle at E and C Bisect GC or EG at H. H is the centre of the required circle.

1867.

1. From A, B as centres and radii equal to double of AB, describe two circles cutting each other at C ABC is the required triangle.

2 Apply Euc I 4

3. Let $ABCD$ be the given square. Bisect AB at E Join ED, EC On the other side of CD describe the triangle DFC equal to the triangle DEC.

1868.

1 Let ABC be a triangle Through C draw DCE parallel to AB. Apply Euc I 29, 13.

2 Let AB, EF be the two sides of the two given squares, also let EF be greater than AB From B draw BC at right angles to AB From the centre A and with radius equal to EF describe a circle cutting BC at C

3. See Addl. Prop. III, p. 187.

4. Apply Euc. I. 32 and III 20

5 (1) Let AB be the given sum and C the hypotenuse. At A in BA make the angle BAD equal to half a right angle. From the centre B and at the distance C describe a circle cutting or touching AD at D. Draw DE perpendicular to AB.

(2) Apply Euc. III. 18 and 14.

1869.

1 Let AB be the given difference On AB describe a segment of a circle containing an angle equal to half the difference of the angles at the base In the circle place AC equal to the given base Produce AB to D At C in BC make the angle BCD equal to the angle CBD ACD is the required triangle.

2 Apply Euc III 36 3. See Ex 11, Book IV p. 358.

1870

1. Apply Euc. I 38

2 Bisect the angle AOB by OE Draw CE at right angles to OA meeting OE at E E is the centre of the required circle.

3 Addl Prop IV p 91

1871

1 Apply Euc I 32, *Cor* 1

2 Let A, B and C be the three straight lines Place A, B at right angles to each other Join the extremities Place C at right angles to the hypotenuse from one of its extremities and draw a second hypotenuse

3 Prove OK equal and parallel to AF, and AKO a right angle (Euc I 33 and 35)

4 Let AB be the given base On AB describe a segment of a circle containing an angle equal to the given vertical angle From A draw AD at right angles to AB and equal to the given perpendicular From D draw DE parallel to AB cutting the segment at G, E Join AE, EB, or join AG, GB Only one triangle will be formed, if GDE touch the circle

5 They meet at the centre of the circumscribed circle. (Euc IV 5)

1872

1 The rectangle contained by the two parts is the greatest possible when they are equal Addl Prop V p 189

2. From P draw PD perpendicular to AB (Euc II 5.)

3 The square on the diameter (or four times the sq on the radius) is equal to the squares of the parts See Ex 84, p. 296.

4. The hexagon is double the triangle

5. Apply Euc I 32, *Cor* 1

1873.

1. Let D, E, F be the middle points of the three sides of the triangle ABC Join DE, EF, DF Produce DF to G, making FG equal to DF Join GC. Apply Euc. I. 38.

2 Draw FK perpendicular to BC, and prove it equal to GF or FH (I 26.)

3 Join BC At B in AB make the angle ABD equal to the angle ABC (I. 23)

1874.

2. Let ABC be the given triangle and the angle at B equal to two-thirds of a right angle Draw AE perpendicular to BC. AB is double of BE Apply Euc II 13

3. (1) The centre of the circle must lie in the straight line bisecting at right angles the line that joins the two given points

(2) The straight lines joining the three points must form a triangle

(3) The quadrilateral formed by joining the four given points must be such that any two of the opposite angles are together equal to two right angles

4 Find the centre O Join BO, DO, EO, CO, FO EO, DO, FO may be proved to be equal to one another etc

5 Find the centre O Join PO, BO and AO (III 20 and I 37)

1875

1. Apply Euc I 29, 26 2 See p 172.

3. Let ABC be the circle DA, DB are tangents drawn from D a point without the circle Find the centre E Join BE and produce it to meet the circle at C Join AC, EA and AB. Apply Euc III 18 and I 32, 5

4 Apply Euc III 35 and I 47

1876

1 Bisect OB at E Draw EF at right angles to AB meeting OD at F Draw FG parallel to AB Apply Euc I 26

2 BAC is a right angle and BC is a diameter of the smaller circle (III 31)

3 BAC is a semicircle, therefore BC is its diameter. Bisect BC at D and AC at E Join DE and produce it to meet the circle at F EF is the height of the arc DF=radius, and $DE=\frac{1}{2}BA$.

1877.

1. Addl Prop VI, p. 281

2. Apply Euc I 32, *Cor* 1 The number of sides is twenty.

1878

1. The contained angle is a right angle. Apply Euc. I 47.

2. See Miscell. Prop. *I* p 384.

3 Addl Prop. II p. 186

4. The square on BL is equal to the rectangle contained by AL, CL (see above) Hence BL is 60 yards PQ is half of AC; See Addl. Prop III and II, p 90

5 (*a*) See Notes on Book II, page 183 (*b*) See Notes on Euc IV. 5 page 342, (*c*) Apply Euc III 35.

1879

1 Apply III 12

2. Miscell Prop I p 384.

3. See Addl Prop VII, p 93

1880

1 Produce AC to E Bisect BC at F, join AF and produce AF to G making FG equal to AF join GC The angle BCE may be proved greater than the angle ABC But BCE is equal to ACD (I 15), therefore ACD is greater than ABC

Otherwise

Draw CE parallel to BA (I 31) The angle ECD is equal to the angle ABC (I 29), therefore ACD is greater than the angle ABC

2 Produce AO, CO to E and F respectively, making EO equal to QB and FO equal to QD Join FE The angle FEO is equal to the angle QBD (I 4), and also equal to the angle ACO A circle described about A, E, F, will also pass through C. Apply III 35

3 The side of the equilateral triangle inscribed in the circle is half of the side of the equilateral triangle described about the circle, &c

1881

1. Let $ABCD$ be the given quadrilateral Join AC, draw DE parallel to AC meeting BC produced at E ABE is the required triangle I 37.

2. The angles ADO, AEO are right angles, therefore a circle described about AO as diameter will pass through the points D and E. The angle DAE is the supplement of the angle DOE (III 22), and therefore it is constant. Therefore DE is constant (III. 26).

1882.

1. $\angle DOB + \angle DOA = 2\angle DOB + 2\angle BOC$.

2. Apply I. 5, 27, 29 and 14.

3. Apply Addl. Prop. III. p 187.

4. Apply III 20 and I. 32.

5. Produce BO to meet the ⊙ at E. $AC.CD=BC.CE$.

$$(AC)^2=(AO)^2+(OC)^2$$

$$\therefore\ AC.CD+(AC)^2=BC.CE+(AO)^2+(OC)^2,$$

$$\therefore\ AC.AD=BC.CE+(OC)^2+(OB)^2=2(OB)^2$$

1883.

1. Apply Euc I 32   2 Apply Euc III 21 and I. 32.

1885

1 Let AB be the given straight line to be divided, and C the other given straight line Draw from B, BD at right angles to AB, making BD equal to C Join AD and make the angle ADE equal to the angle BAD Then AE and EB are the required parts

2. Join AD Apply Euc III 31, and I 32, 5

3 The three straight lines, when produced, will meet to form a triangle. For the first part, apply Euc. IV 4 As for the second part, apply Addl Prop XIX p 104 Thus there may be three escribed and one inscribed circles

4 See Obs to Prop D and E, pp 376, 377, and Miscell Prop. I p. 384

1886.

1. Same as I 33   2 Same as II 7

3 (a) The demonstration is the same as that of III. 11.
(b) Addl. Prop IV. p 279

4. Inscribe a triangle in the given circle equiangular to the given isosceles triangle Divide each of the base angles into seven equal angles, making each of these angles equal to the vertical angle. &c. &c

5. Apply Ex 4, Prop 20, Book I

6 Let O and Q be the centres of the circles ABD and ABF respectively. Join OQ cutting AB at P OQ is bisected at right angles by AB. Draw OR perpendicular to CD and QS to EF. $OR=OP=QP=QS$ (III 14) We can prove $PH=RH=SH$. $\angle OHR=\angle OHP=\angle QHP=\angle QHS$.

$\therefore\ \angle RHP=\angle PHS$.

1887.

1. Through F draw GFH parallel to BC cutting BA and CD at G and H respectively $AG=DH$, &c.

2 Apply Euc. III. Prop 36.

3. Let E be the point on the same side of BC as A. E is equidistant from AB and AC. Also because the angle $ACE = \angle CBE$ (III 32) $= \angle BCE$ $\therefore$ E is equidistant from BC, AC $\therefore$ E is the centre of the circle which touches AB, BC and AC. Produce AC to G, $\because$ the angle $FCG = \angle FBC$ (III 32) $= \angle FCB$.

$\therefore$ BC, and AC produced, are equidistant from F, likewise BC, and AB produced are equidistant from F $\therefore$ a circle whose radius is FD and centre is F will touch BC, and AB, AC produced

1888

1 See Ex 2 Prop 33, Book I
2 See Ex 1, Prop 17, Book III
3 Join BG, DG, EG, CG, FG

The angles ABC, ACB together make up the supplement of $\angle A$

But $\angle ABC = \angle DGE$, and $\angle ACB = \angle$s CEF, CFE
$= \angle$s DEB, $EGC = \angle$s DGB EGC

Hence the sum of the angles DGE, EGC and DGB, or the angle BGC, is the supplement of $\angle A$

$\therefore$ a circle may be described about $ABGC$ (Addl. Prop III p 348) &c

1889

1 See Ex 5 Prop 47, Book I

2 Let ABC be the given equilateral triangle Draw AD perpendicular to BC Complete the rectangle $BDAE$ Produce BD to F making $DF = AD$ On BF as diameter describe a circle cutting AD and AD produced at G and H The square on GD is equal to the equilateral triangle ABC

3. Let ABC be a triangle Describe a circle about ABC Draw $AD \perp BC$ Produce AD to meet the circle at E Make $DO = DE$ Join BO and produce it to meet AC at F Join BE. $\because$ in the $\triangle$s BED, BOD $DO = DE$, BD is common, and the angle BOD = angle BED, $\therefore$ $\angle DBO = \angle DBE$ But $\angle DBE = \angle DAC$, (III 21) $\therefore$ $\angle DBO = \angle OAF$, and the angle BOD = angle AOF. $\therefore$ the angle ODB = the angle OFA = a right angle, &c.

1890

1. Deduce from Addl Prop IV p 91

2. Let the diagonals of the quadrilateral $ABCD$ intersect at O; produce OC, OD to E and F respectively, making $CE = AO$, and $DF = BO$

$\triangle ABO = \triangle ADF$, and $\triangle BOC = \triangle DCF$, and
$\triangle AOF = \triangle CEF$.(I 38.) &c.

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